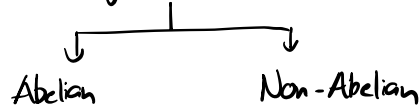


FADDEV-POPOV QUANTISATION.

- Deal w/ redundancy in Gauge theories. (Recall for $U(1)$ what's that)



↳ Both have same procedure to be quantized, but Abelian MUCH EASIER.
(We will soonish see why!)

Start Quantization for $U(1)$ gauge boson.

- [Euclidean space]: Go from $(M) \rightarrow (\mathbb{R}^4 \equiv E)$ by a Wick rotation (Essentially a COV)

$$x^0 = x_0 = t = -ix_4 = -ix^4, \quad \partial_0 = \frac{\partial}{\partial x^0} = i \frac{\partial}{\partial x_4} = i\partial_4$$

↳ In Quantizing $U(1)$ gauge boson, we will deal w/ A_μ , which transforms like x_μ . So we also rotate A_0 like $x_0 = -t$: $A_0 \equiv iA_4$

$$\begin{aligned} \hookrightarrow A_\mu = (A_0, A_j) &= (iA_4, A_j), \quad A_\mu A^\mu = -A_0^2 + A_j^2 \\ &= -(iA_4)^2 + A_j^2 \\ &= A_4^2 + A_j^2 \end{aligned}$$

- [E space in E_i]:

$$\begin{aligned} E_i^{(M)} = F_{0i} &= \partial_0 A_i - \partial_i A_0 = i\partial_4 A_i - \partial_i (iA_4) \\ &= i(\partial_4 A_i - \partial_i A_4) \\ &= i F_{4i} \\ &= i E_i^{(E)} \end{aligned}$$

$$\text{Similarly, } F^{0i} = -iF^{4i}$$

- [E space massless $U(1)$ action]

{ Why not transform this? Wick rotation does $M \rightarrow E$ transformation by just COV in zeroth component }

$$\begin{aligned} \mathcal{L}_{EM} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{0i} F^{0i} - \frac{1}{4} F_{ij} F^{ij} = -\frac{1}{4} F_{4i} F^{4i} - \frac{1}{4} F_{ij} F^{ij} \\ &= -\frac{1}{4} F_{\mu\nu}^{(E)} F^{(E)\mu\nu} \end{aligned}$$

↳ This is just the $\mathcal{L}$, we need to see the action

$$\underbrace{i \int d^4x \mathcal{L}_{EM}^{(E)}}_{-i \int d^4x E} = \frac{1}{4} F_{\mu\nu}^{(E)} F^{(E)\mu\nu} = \frac{1}{2} [(E_i^{(E)})^2 + (B_i^{(E)})^2]$$

(Only $dx^0 \rightarrow -i dx^4$) $(= \frac{1}{2} [(B_i)^2 - (E_i)^2])$

* We will drop the (E) index now, but will be working in E (Till I say we change)

• [The Action and the path integral] : (Integrating by parts)
↳ GAUGE FIXING idea

$$S_{EM}[A] = \int d^4x \left[-\frac{1}{2} A_\mu (\partial_{\mu\nu}^2 - \partial_\nu \partial_\mu) A_\nu \right] \rightarrow \text{This is the Euclidean action.}$$

$$Z = \int DA_\mu(x) e^{-S_{EM}[A]} \quad \text{--- ①}$$

(i) $A'_\mu \neq A''_\mu - \frac{1}{e} \partial_\mu \chi$

(ii) $A'_\mu = A''_\mu - \frac{1}{e} \partial_\mu \chi$

Remind everyone where this comes from.

Recall the idea of Path $\int$
:- You are integrating over all the possible configuration $A_\mu(x)$ 

- The sum of all these gives you the probability amplitude.
- Classically, the one minimizing the action is the only one contributing.

↳ In eq(1), there are two types of contributions to $A_\mu(x)$ [The function over which we are taking the path $\int$]

(i) This is the desired one as we are $\int$ over all physically inequivalent configs of A_μ that give the quantum behaviour of the field.

(ii) This is (i) + physically equivalent configs of A_μ

↳ Not good! As we have a copy of A_μ for EACH choice of χ which end up in the $\int$.

↳ Try 'fixing' this : $\int DA_\mu(x) = \underbrace{\int d\chi}_{\text{over Gauges}} \underbrace{\int DA_\mu^{(GF)}(x)}_{\text{over physically relevant fields (GF)}} \quad \text{--- ②}$

↳ Gauge-fixed.

- (Doing this is a challenging task - even for Abelian, but much more for Non-abelian)
- (The whole business of GF can be ^{naively} boiled down to : Avoid double-counting)

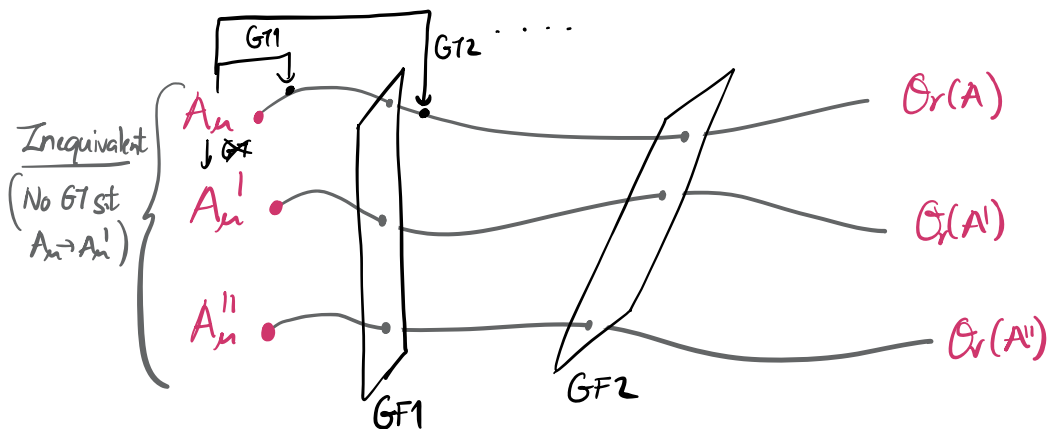
↳ In eq.(2), if we accomplish doing this, we remove all dependence of x from the path $\int$

↳ The $\int dx$ just becomes a multiplicative factor in $\mathbb{Z}$. (The interpretation is that : This $\int$ value is the volume of the internal space defined by $U(1)$)

- We know choosing $\partial_\mu A_\mu = 0$ (Lorenz gauge) removes the redundancy. Our goal is to see why does it do that!

↳ Start w/ a more general gauge : $\partial_\mu A_\mu = C(x)$ — (3)

For a given specific field configuration A_μ , the **orbit** $Or(A)$ is the set of all other configurations that can be obtained through A_μ (through GT)



↳ In the 'space of all possible' GF conditions, the 'good' are the ones that only have one point of intersection between this 'space' & $Or(A)$ (for each ineq. config)

↳ For Abelian theories, we can assume this intersection is unique (for perturbative regime)

For Non Abelian, there is a problem of Gribov copies (Very hard to get 1 intersection per orbit)

(We will cross this bridge when we come to it, right now let's focus on quantizing $U(1)$)

↳ GF is finding these single intersection planes, & $\int dx$ is the volume you get by $\int$ of these planes.

• (PARAMETRIZE the orbit)

Assume that we are travelling through the orbit using the following transformation

$$A_\mu(x) \rightarrow {}^x A_\mu(x) \equiv A_\mu(x) + \partial_\mu x(x) \rightarrow {}^x A_\mu(x) \text{ is any point in the orbit that I can get starting}$$

Taking ∂^μ , $\partial^\mu \chi A_\mu(x) = \partial^\mu A_\mu(x) + \partial^2 \chi(x)$ from $A^\mu(x)$
 $Z \rightarrow C(x)$

$$\partial^2 \chi(x) = -\partial^\mu A_\mu(x) + C(x)$$

Basically, this is the value of χ that brings us into GF condition.
 $\partial^\mu A_\mu = C$

This is the transformation we need such that eq.3. is satisfied.

• [PROVE IDENTITY]

Use the two variables $\chi A_\mu(x)$, $\chi(x)$ to prove the following identity:

$$\int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta(-\partial_\mu(\chi A_\mu(y)) + C(y)) = \frac{1}{\text{Det}(-\partial^2)}$$

Why? Because if this is true, we will have found the identity

$\int d\chi$ ^{Integrating over all the gauges.} $\delta[-\partial_\mu A + C]$: δ functional s.t. the derivative of A , must be C $\forall y \in \mathbb{R}^4$

$$1 = \text{Det}(\partial^2) \int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta(-\partial_\mu(\chi A_\mu(y)) + C(y))$$

(F)

If we place this identity inside our path integral of A , through δ we can impose eq.3 ($\partial_\mu A_\mu = C(x)$)

Let us demonstrate the above point:

It is a const. $\cdot$

$$-\partial_\mu(\chi A_\mu) + C = -\partial^2 \chi - \partial_\mu A^\mu + C = -\partial^2 \chi + \overset{\text{into}}{\partial^2 \chi^{(A)}}$$

This is the function that puts us in the point in the χ space that crosses the GF condition for a value of C .
IMP

Make a change of variables: $\chi \rightarrow \chi - \chi^{(A)}$ (Inside the integral)

$$\int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta[-\partial_\mu(\chi A_\mu(y)) + C(y)] = \int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta[-\partial^2 \chi(y)]$$

$G(\chi A_\mu)$ No change in the measure.

$$G(\chi A_\mu(y)) = -\partial_\mu(\chi A_\mu(y)) + C(y) = -\partial^2 \chi(y)$$

⊕

$$\frac{\delta G(x A_n(y))}{\delta x(x)} = -\partial^2 \delta(x-y) \equiv -\partial^2(x,y)$$

↙ This op is acting on $x(y)$ inside the δ function.

↪ This op is like a Jacobian for a change of variables:

$$x \rightarrow G(x A_n) \quad (\text{Let's go to the discrete version to see this})$$

$$\begin{aligned} \int \prod_{i=1}^n dx_i \prod_{j=1}^N \delta(\Delta_{ij} x_j) &= \int d^n x \underbrace{\delta^n(\Delta x)}_{\equiv \delta^n(\vec{n}), \quad n_i = \Delta_{ij} x_j} \\ &= \int \frac{d^n n}{\text{Det}(\Delta)} \delta^n(\vec{n}) \\ &= \frac{1}{\text{Det}(\Delta)} \underbrace{\int d^n n \delta^n(\vec{n})}_{=1}, \text{ just } \int \text{over } \delta^n. \\ &= \frac{1}{\text{Det} \Delta} \end{aligned}$$

↳ The functional version:

$$\begin{aligned} \int \prod_{x \in \mathbb{R}^d} dx \prod_{y \in \mathbb{R}^d} \delta(-\partial^2 x(y)) &= \int x \frac{dn(x)}{\text{Det}[\frac{\delta G}{\delta x}]} \prod_y \delta(n(y)) \\ &= \frac{1}{\text{Det}[\dots]} \int \prod_x \frac{d^n n}{\text{Det}(\Delta)} \prod_y \delta(n(y)) \\ &= \frac{1}{\text{Det}[\dots]} \end{aligned}$$

$n(y) = -\partial^2 x(y)$
 $dn = \text{Det}(-\partial^2) dx$
 $\hookrightarrow = \text{Det}[\frac{\delta G}{\delta x}]$

$$\therefore \int \prod_x dx \prod_y \delta(-\partial^2 x(y)) = \frac{1}{\text{Det}[-\partial^2]} = \frac{1}{\text{Det}[\frac{\delta G}{\delta x}]}$$

- If I plug this identity inside my path $\int$, I would have GF^d it.
- What is the effect of this Dirac δ that I am putting in, in terms of a Lagrangian term.
 - ↳ Can I convert this to a term in the Lagrangian to see physically the effect?
 - ↳ Need one more identity:

→ The $\int$ is a Gaussian.

$$N(x) \int Dc \underbrace{e^{-\frac{1}{2\alpha} \int dx \dot{c}^2(x)}}_{\text{Just a \#}} = 1$$

Normalization,
i.e. $\frac{1}{\int \dots}$

- We have two identities = 1. Lets multiply them:

$$N(x) \int Dc \underbrace{e^{-\frac{1}{2\alpha} \int dx \dot{c}^2(x)}}_{\text{Just a \#}} \text{Det}(-D^*) \int Dx S[-\partial_\mu^x A_\mu + c] = 1$$

- Now we use the δ to do the path $\int$ in c .

$$\int Dx N(x) e^{-\frac{1}{2\alpha} \int dx (\partial_\mu^x A_\mu(x))^2} \text{Det}(-D^*) = 1$$

- Insert this identity for the path $\int$ in field A .

$$\int DA_\mu e^{-S[A]} \underbrace{O[A]}$$

Some observable/operator → Anything that I can calculate in this theory.

$$= \int DA_\mu e^{-S[A]} O[A] \int Dx \underbrace{N(x)}_{\text{Just a \#}} e^{-\frac{1}{2\alpha} \int dx (\partial_\mu^x A_\mu(x))^2} \underbrace{\text{Det}(-D^*)}_{\text{Just a \#}}$$

- Few interesting things that I can do.
- (Ft, VU IMP) : Det does not depend on x or A . (See this from ⊕) $\frac{\delta G}{\delta x}$ does not depend on A .

↳ This is the point where if we had non-abelian theory, would be different.

In non-abelian theories $\frac{\delta G}{\delta x}$ will depend on A . (Need extra steps to deal w/ this determinant)

$$\int DA_n e^{-S[A]} \mathcal{O}[A] = \text{Det}(-\delta) N(x) \int dx \int DA_n e^{-S[A] - \frac{1}{2\alpha} \int d^4x \underbrace{(\partial_\mu x A_\mu(x))^2}_{\oplus}} \mathcal{O}[A]$$

• Basically, I am calculating the observable w/ this new action.

• Bit nicer : $x A = A + \partial_\mu x$ or $\underline{A} = x A - \partial_\mu x$
Origin of orbit.

• Major point: Action is Gauge invariant. So we can remove or add x at will.
 $S[A] = S[\underline{A}]$ & $\mathcal{O}[A] = \mathcal{O}[x A]$

• Big assumption: $\int^n$ measure is also Gauge invariant. $DA = D^x A$.
(Neither guaranteed, nor obvious)
↳ True & can be proved for most Gauge symmetries.
↳ Few where these fail, you get Anomaly.
↳ QED not anomalous so it is true & can be proved.

⇒ Bring in x indices everywhere where invariant. Then remove from everywhere, especially $\oplus$ where its not invariant.

$$\int DA_n e^{-S[A]} \mathcal{O}[A] = \underbrace{\text{Det}(-\delta) N(x)}_{\text{other \#s.}} \underbrace{\int dx}_{\downarrow \text{Just a \# / Big Volume.}} \int DA_n e^{-S[A] - \frac{1}{2\alpha} \int d^4x (\partial_\mu A_\mu(x))^2} \mathcal{O}[A]$$

• Expectation value will be written.

$$\langle \mathcal{O}[A] \rangle = \frac{\int DA \mathcal{O}[A] e^{-S[A]}}{\int DA e^{-S[A]}} \frac{\uparrow}{\uparrow} = \frac{\# \int DA \mathcal{O}[A] e^{-S_{\text{eff}}[A]}}{\# \int DA e^{-S_{\text{eff}}[A]}}$$

• Now $\mathcal{L}_{\text{eff}} = \frac{1}{2} F_{\mu\nu}^2 + \underbrace{\frac{1}{2\alpha} (\partial_\mu A_\mu)^2}_{\text{Gauge fixing term. } \mathcal{L}_{\text{GF}}}$

↳ You could just add in this term by hand saying you are not really changing the classical theory, but maybe you are changing the quantum theory. → We showed that we are not.

↳ Assumption was $DA = DA^x$. There will be anomalous theories → Symmetry of classical but not Quantum.

Photon propagator

$$S_{\text{EFF}} = \int d^n \left[-\frac{1}{2} A_\mu (\delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A_\nu \right] - \underbrace{\frac{1}{2\alpha} A_\mu \partial_\mu \partial_\nu A_\nu}_{\text{GF part}}$$

$$= \frac{1}{2} \int d^n A_\mu(x) \underbrace{\left(-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right)}_{\substack{\text{Invertible.} \\ (G^{(0)})^{-1}_{\mu\nu}}} A_\nu(x)$$

$$\begin{aligned} &= \frac{1}{2} \int d^n \int \frac{d^4 k}{(2\pi)^4} \tilde{A}_\mu(k) \underline{e^{ikx}} \left(-\delta_{\mu\nu} \partial^2 + \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right) \int \frac{d^4 k'}{(2\pi)^4} \tilde{A}_\nu(k') \underline{e^{ik'x}} \\ &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4} (2\pi)^4 \delta^4(k+k') \tilde{A}_\mu(k) \left(\delta_{\mu\nu} k'^2 - \left(1 - \frac{1}{\alpha} \right) k'_\mu k'_\nu \right) \tilde{A}_\nu(k') \\ &\quad \text{Then } k' \rightarrow -k \\ &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} A_\mu(-k) \underbrace{\left(\delta_{\mu\nu} k^2 - \left(1 - \frac{1}{\alpha} \right) k_\mu k_\nu \right)}_{(G^{(0)})^{-1}_{\mu\nu}(k)} A_\nu(k) \end{aligned}$$

$$\left(\delta_{\mu\nu} k^2 - \left(1 - \frac{1}{\alpha} \right) k_\mu k_\nu \right) G^{(0)}_{\mu\nu}(k) = \delta_{\mu\nu}$$

$$G^{(0)}_{\mu\nu}(k) = \frac{1}{k^2} \left(\delta_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) \frac{k_\mu k_\nu}{k^2} \right)$$

Photon prop in euclidean mom space.

Depends on α . Related to GF in a complicated way

$\alpha = 1$ Feynman gauge.

We did $\partial_\mu A_\mu = C$ & then $\int Dc \ e^{-\frac{1}{2\alpha}} (\partial^\mu c)^2$ 