VERTEX CORRECTION & e^- g-factor.

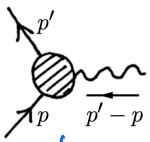
- Magnetic moment operator for e^- : $\hat{\vec{\mu}} = g\left(\frac{Q|e|}{2m}\right)\hat{S}$ (Q = -1)

↳ Classically : g of a charged spinning object , $g=1$

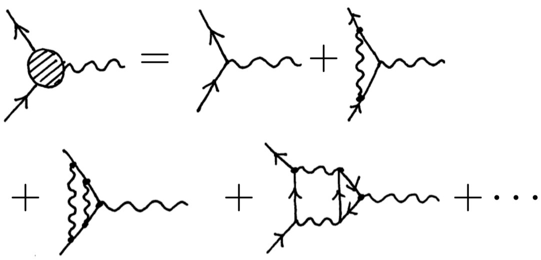
↳ RQM : Using Dirac eqn , $g=2$

↳ QFT : Loop corrections (1st order) , $g = 2 + \frac{\alpha}{\pi}$

↳ Fact that e^- can emit virtual photons, changes the form of $e^-\gamma$ interaction measurably.



$\equiv \; -iQ|e|\tilde{\Gamma}^{\mu}(p,p') = \sum \left(\begin{array}{c} \text{All amputated insertions with one} \\ \text{incoming fermion line, one outgoing} \\ \text{fermion line and one photon line.} \end{array} \right).$



Vertex function can be interpreted as : 'Off shell photon decaying into e^-e^+ pair'

- This vertex function can be interpreted as off-shell photon decaying to e^-e^+ pair.

Facts :		Charge	Parity	L	S
	γ	0	-1	0	1
	e^-	-1	$+1$		$\frac{1}{2}$
	e^+	$+1$	-1		$\frac{1}{2}$

(For e^-e^+)

- Take , $L=0$, $S=0$ (Singlet) $J=0$, ✗
- Take , $L=0$, $S=1$ (Triplet) $J=1$ ✓ , Parity? $\underset{e^-}{(+1)}\,\underset{e^+}{(-1)}\,\underset{(-1)^L}{(-1)^0}\,\underset{\substack{S=1 \\ \text{is symmetric}}}{(+1)} = -1$ ✓
- Take , $L=1$, $S=1$ $J=0,2$ ✗

• Take, $L=2$, $S=1$ $J=1,3$ ✓, Parity $(+1)(-1)(-1)^2(+1) = -1$ ✓

Seems to be a pattern, L even for $S=1$, but for any even $L>2$, $J \neq 1$ when $S=1$.

$\therefore$ Two allowed states for $e^- e^+$ pair, $L=0$, $S=1$

$L=2$, $S=1$.

• Vertex function can be written as a sum of two terms,

$$\tilde{\Gamma}^\mu(p,p') = \gamma^\mu F_1(\overset{(q^2)}{q}) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(\overset{(q^2)}{q})$$

Dirac form factor

Pauli form factor.

$q^\mu = p'^\mu - p^\mu$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

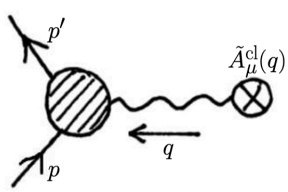
[7] In fact, $F_1(q)$ may be thought of as a Fourier transform of the charge distribution and has the property that $F_1(0) = 1$ at all orders of perturbation theory.

For vanishing q, we only have first order contribution,

$$\tilde{\Gamma}^\mu = \gamma^\mu \Rightarrow \begin{aligned} F_1(0) &= 1 \\ F_2(0) &= 0 \end{aligned}$$

We will now sketch how we calculate the g-factor.

Step 1 : Consider interaction vertex coupled to a classical source $A_\mu^{cl}(x)$



$$\equiv -iQ|e|\tilde{\Gamma}^\mu \tilde{A}_\mu^{cl}(\underline{q})$$

Momentum of photon.

Couple this to the spin or current of the incoming and outgoing electron, we get the amplitude,

$$i\mathcal{M} = -iQ|e|\bar{u}(p')\tilde{\Gamma}^\mu(p,p')u(p)\tilde{A}^{\rm cl}_\mu(\underset{p'-p}{q}) \qquad \text{(Suppressed } \delta \text{ fn)}$$

Inserting the two form factor form of the vertex function,

$$i\mathcal{M} = -iQ|e|\bar{u}(p')\left[\gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q)\right]u(p)\tilde{A}^{\rm cl}_\mu(q).$$

Step 2 : Take the $\bar{u}'\gamma^\mu u$ from the first term and put in the Gordon decomposition

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

This eliminates the γ^μ , making the first term spin independent & second term spin dependent.

Plugging this into the Feynman amplitude yields,

$$\begin{aligned} i\mathcal{M} &= -iQ|e|\bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m}\right)u(p)\tilde{A}^{\rm cl}_\mu(q)F_1(q) \\ &\quad + \frac{Q|e|}{2m}\left[\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)\right]\tilde{A}^{\rm cl}_\mu(q)\left\{F_1(q) + F_2(q)\right\}. \end{aligned}$$

This makes it into such a form that we have information on the g-factor in the second term

Step 3 : Take the non-relativistic limit $u(p) \approx \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}$

And, we need the explicit forms of the components of $\sigma^{\mu\nu}$

$$\sigma^{0i} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix} \quad \text{and} \quad \sigma^{ij} = \begin{pmatrix} \varepsilon^{ijk}\sigma^k & 0 \\ 0 & \varepsilon^{ijk}\sigma^k \end{pmatrix}$$

Using this we can show $\bar{u}'\sigma^{0i}u = 0 \qquad \bar{u}'\sigma^{ij}u = 2m\left[\xi'^\dagger\sigma^k\xi\right]\varepsilon^{ijk}$

Put all this together for $q \to 0$ we then have the spin dependent amplitude as,

$$\begin{aligned} \begin{matrix}\text{Spin-dependent} \\ \text{amplitude}\end{matrix} &= (2m)\frac{Q|e|}{2m}\left[\xi'^\dagger\sigma^k\xi\right]\varepsilon^{ijk}q^j\tilde{A}^{{\rm cl}\,i}(q)\left\{F_1(0) + F_2(0)\right\} \\ &= (2m)\frac{iQ|e|}{2m}\left[\xi'^\dagger\sigma^k\xi\right]\tilde{B}^k(\boldsymbol{q})\left\{1 + F_2(0)\right\}, \end{aligned}$$

where we have assumed $A^{cl}(x) = (0, \vec{A}^{cl}(\vec{x}))$

$$\Big\downarrow \text{Momentum space}$$

$$\tilde{B}^k(\vec{q}) = i\epsilon^{ijk}q^i\tilde{A}^{cl\,j}(\vec{q})$$

Finally, looking at the square brackets in the expression as $\langle\hat{\vec{S}}\rangle = \dfrac{\xi'^\dagger\vec{\sigma}\xi}{2}$,

Giving us the spin dependent part of the amplitude to be proportional to $Q|e|\{1+F_2(0)\}\langle\hat{\vec{S}}\rangle\cdot\vec{B}$

Now implement Born approximation again, $i\mathcal{M} = -i\langle f|V(x)|i\rangle$

(going from relativistic normalization to non-relativistic case, we have to make a correction by dividing by twice the energy of the electron that is scattered $= 2m$

We will get the real-space potential,

$$V(\boldsymbol{x}) = -\frac{Q|e|}{m}\{1 + F_2(0)\}\langle\hat{\boldsymbol{S}}\rangle \cdot \boldsymbol{B}(\boldsymbol{x}).$$

$$V(\boldsymbol{x}) = -\langle\hat{\boldsymbol{\mu}}\rangle \cdot \boldsymbol{B}(\boldsymbol{x}) = -\left[g\left(\frac{Q|e|}{2m}\right)\langle\hat{\boldsymbol{S}}\rangle\right] \cdot \boldsymbol{B}(\boldsymbol{x}),$$

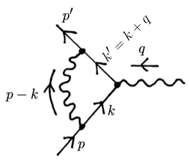
We can now read off, $g = 2\,[1 + F_2(0)]$

This shows that g is not exactly 2, but the correction is determined by the Pauli form factor $F_2(0)$

Step 4 : Now, we can actually compute the correction to the g factor by evaluating its contribution to $\tilde{\Gamma}^\mu$.

$F_1(0) = 1$, $F_2(0) = 0$

- First order we already know
- Second order there are no diagrams
- Third order we have



From the labelled diagram, shown in Fig. 41.10, we can write down the amplitude for the vertex as

$$\tilde{\Gamma}^\mu_{\text{diagram}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-\mathrm{i}g_{\nu\rho}}{(p-k)^2 + \mathrm{i}\epsilon}\bar{u}(p')(\mathrm{i}|e|\gamma^\nu)\frac{\mathrm{i}}{\not{k}' - m + \mathrm{i}\epsilon}\gamma^\mu\frac{\mathrm{i}}{\not{k} - m + \mathrm{i}\epsilon}(\mathrm{i}|e|\gamma^\rho)u(p). \tag{41.39}$$

This is divergent, so we must also include a third-order counterterm. With the inclusion of the counterterm in Fig. 41.1(e) and two pages or so of Dirac algebra[8] one can derive a result for the Pauli form factor which is $F_2(0) = e^2/8\pi^2$.

Recall that in the units used here, the fine structure constant is given by $\alpha = e^2/4\pi$. Because $\alpha = 1/137.036...$ we have $g = 2.0023...$ In SI units, $\alpha = e^2/4\pi\epsilon_0\hbar c$.

RESULT : $$g = 2\left(1 + \frac{\alpha}{2\pi} + \ldots\right).$$

All detailed calculations can be found in Schwartz, chapter 17.