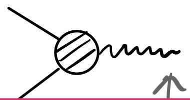
- [Two beautiful results] of QED can be revealed by considering e^-e^+ loops.
  - ↳ These loops contain divergences that can be tamed using RN.

(Result 1) Electronic charge is not constant

(Result 2) g factor of electron is not quite $g=2$.

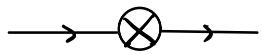
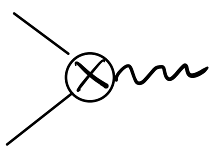
- [Three Greens required to RN QED]

(i) Electron 1PI self-energy		:	$-i\tilde{\Sigma}(\not{p})$
(ii) Photon 1PI self-energy		:	$i\tilde{\Pi}^{\mu\nu}(q)$
(iii) Interaction Vertex function		:	$-iQ\lvert e\rvert\tilde{\Gamma}^{\mu}(p,p')$

As depicted here

These f^{ns} are an ∞ sum of Feynman diagrams

- [Counterterms]
  - ↳ While computing each of these diagrams, we will have divergences.
  - ↳ By adding CTs : (i) Shift from bare to phys parameters
    (ii) Remove divergences.
  - ↳ QED is RN^{able} & we need three CT's (One corresponding to each above)

(CTi) Electron self-energy		$i(\not{p}B+A)$
(CTii) Photon self-energy		$i(g^{\mu\nu}q^2-q^{\mu}q^{\nu})C$
(CTiii) Interaction vertex		$iQ\lvert e\rvert\gamma^{\mu}D$

- [Impose RN conditions]

↳ In the prev. tutorial, we talked about two ways to RN a theory.

We chose to use RN conditions :

$-i\tilde{\Sigma}(\not{p}=m)=0$	Fix elec. mass to m
$i\tilde{\Pi}^{\mu\nu}(q=0)=0$	Fix photon mass to 0
$-iQ\lvert e\rvert\tilde{\Gamma}^{\mu}(p'-p=0)=iQ\lvert e\rvert\gamma^{\mu}$	Fix charge to $Q\lvert e\rvert$

Step 1: Examine $\Pi^{\mu\nu}$ & its effect on the dielectric vacuum.

- [Recall: 'Dielectric']
  ↳ Dielectric is an insulator that may be polarized with $\vec{E}$ applied.
  ↳ Encoded in ϵ, which tells us how much does it alter the electrostatic potential of the charge. $V(r) \sim \frac{e}{4\pi\epsilon_0}\frac{1}{r}$

- [Why are we talking about dielectric?]
  ↳ RN'd QED predicts that the interacting vacuum is a dielectric!

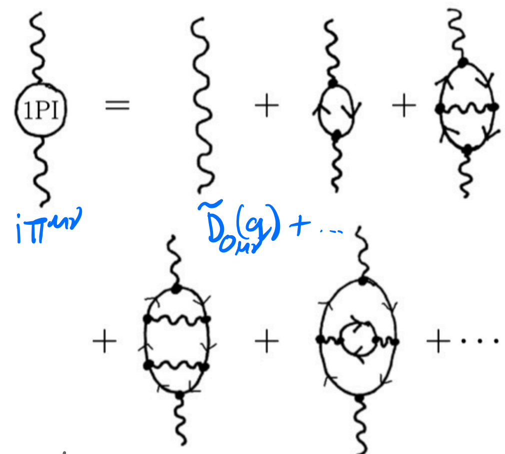
Huh?

↳ This dielectric nature arises because: 'Turning on interactions in our theory, dresses the bare photon in a cloud of e^-e^+ pairs'

(Why do I say 'Turning on interactions in our theory' – Because experimentally, if our theory is correct, we cannot really turn off these interactions. This is the whole reason why we RN in the first place)

↳ Lets see how this works,

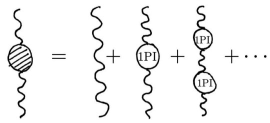
$$i\tilde{\Pi}^{\mu\nu}(q) = \sum\begin{pmatrix}\text{All 1PI diagrams inserted} \\ \text{between the photon lines}\end{pmatrix}$$

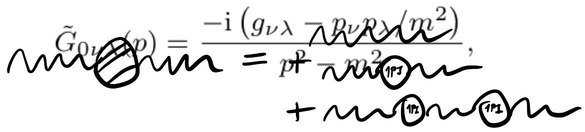


↳ Various ways of the photon removing e^-e^+ pairs from vacuum & returning them.

- Using the defⁿ of $i\Pi^{\mu\nu}$ & the free propagator $\tilde{D}_{0\mu\nu}$, we can compute the full interacting propagator $\tilde{D}_{\mu\nu}$

- [Dyson series]





In equations the sum is

$$\begin{aligned}\tilde{D}_{\mu\nu}(q) &= \tilde{D}_{0\mu\nu}(q) + \tilde{D}_{0\mu\lambda}(q)[\mathrm{i}\tilde{\Pi}^{\lambda\eta}(q)]\tilde{D}_{0\eta\nu}(q) + \dots \\ &= \frac{-\mathrm{i}g^{\mu\nu}}{q^2} + \left(\frac{-\mathrm{i}g_{\mu\lambda}}{q^2}\right)\mathrm{i}\tilde{\Pi}^{\lambda\eta}(q)\left(\frac{-\mathrm{i}g_{\eta\nu}}{q^2}\right) + \dots \\ &= \frac{-\mathrm{i}g_{\mu\nu}}{q^2} + \left(\frac{-\mathrm{i}}{q^2}\right)\mathrm{i}\tilde{\Pi}_{\mu\nu}(q)\left(\frac{-\mathrm{i}}{q^2}\right) + \left(\frac{-\mathrm{i}}{q^2}\right)\mathrm{i}\tilde{\Pi}^{\eta}_{\mu}(q)\left(\frac{-\mathrm{i}}{q^2}\right)\mathrm{i}\tilde{\Pi}_{\eta\nu}(q)\left(\frac{-\mathrm{i}}{q^2}\right) + \dots\end{aligned} \tag{41.5}$$

Summing the series we obtain

$$\tilde{D}_{\mu\nu}(q) = -\frac{\mathrm{i}g_{\mu\nu}}{q^2} + \frac{1}{q^2}\tilde{\Pi}_{\mu\eta}(q)\tilde{D}^{\eta}_{\nu}(q), \tag{41.6}$$

or

$$\left(q^2 g_{\mu\eta} - \tilde{\Pi}_{\mu\eta}(q)\right)\tilde{D}^{\eta}_{\nu}(q) = -\mathrm{i}g_{\mu\nu}. \tag{41.7}$$

Things are simplified rather by the use of the Ward identity, which guarantees that $q_\nu\tilde{\Pi}^{\mu\nu}(q) = 0$. As a result, we can write the photon self-energy as

$$* \quad \tilde{\Pi}^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\tilde{\Pi}(q). \tag{41.8}$$

Substitution of this new form yields

$$q^2[1 - \tilde{\Pi}(q)]D_{\mu\nu}(q) + \tilde{\Pi}(q)q_\mu q_\eta D^{\eta}_{\nu}(q) = -\mathrm{i}g_{\mu\nu}, \tag{41.9}$$

whose solution is

$$\tilde{D}_{\mu\nu}(q) = \frac{-\mathrm{i}g_{\mu\nu}}{q^2[1 - \tilde{\Pi}(q)]} + \frac{\mathrm{i}q_\mu q_\nu\tilde{\Pi}(q)}{q^4[1 - \tilde{\Pi}(q)]}. \tag{41.10}$$

In practical calculations the $q_\mu q_\nu$ part never contributes to any observable amplitude. This is because $\tilde{D}_{\mu\nu}(q)$ is always coupled to conserved currents and we know that $q_\mu J^{\mu}_{\mathrm{em}} = 0$.

*1) An excerpt from Peskin 7.5, "The only tensors that can appear in $\Pi^{\mu\nu}(p)$ are $g^{\mu\nu}$ & $p^\mu p^\nu$

2) The Ward identity tells us $p_\mu \Pi^{\mu\nu}(p) = 0$

3) Implying, $\Pi^{\mu\nu}(p) \sim \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$

→ Only vector on which the problem depends : p^μ, $g^{\mu\nu}$ is the invariant tensor

$\therefore$ $\Pi^{\mu\nu}(p) = (A g^{\mu\nu} + B p^\mu p^\nu)\Pi(p)$, $\quad p_\mu \Pi^{\mu\nu} = p_\mu(A g^{\mu\nu} + B p^\mu p^\nu)\Pi(p) = 0$

$\Rightarrow \quad A = 1 \;\&\; B = -\frac{1}{p^2}$

'Projector' $\forall p_\mu$, $\quad p_\mu = \frac{k_\mu k_\nu}{k^2}p^\nu + \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)p^\nu$

p longitudinal k ('in direction') $\quad$ p transverse k ('Orthogonal dir^n') $\Rightarrow$ Check, if $k_\nu = p_\nu$, $\frac{k^\mu k^\nu}{k^2} k_\nu = k^\mu$

$\Rightarrow$ Resulting renormalized propagator is given by

$$\tilde{D}_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2[1-\tilde{\pi}(q)]}$$

- Pretty neat, all corrections from loops absorbed in
- As it contains loops, it will diverge unless we put in CTs.

Example: (At first order correction, we just have to worry about one diagram)

↳ Notation ($\Pi \to \pi$: $\quad i\pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\tilde{\pi}(q)$)

↳ To ensure that the loop inside $\tilde{\pi}(q)$ doesn't cause any trouble, include the counterterm,

$$i(g^{\mu\nu}q^2 - q^\mu q^\nu)\, C^{(2)}$$ → In eq 41.8 above

↳ Summing the insertion to all orders we will get:

$$\tilde{D}_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2[1-(\tilde{\pi}(q)+C^{(2)})]}$$

↳ Impose RN condition: $\Pi^{\mu\nu}(q)$ vanishes at $q^2=0$, which implies,

$$\tilde{\pi}(q=0) = -C^{(2)}$$

$$\therefore \quad \tilde{D}_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2[1-(\tilde{\pi}(q)-\tilde{\pi}(0))]}$$

↳ This is good enough to remove divergences from $\tilde{D}_{\mu\nu}(q)$ upto second order.

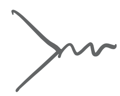
- So whats the story here?
  - ↳ Bare photon tearing e^-e^+ pairs from vacuum.

$\downarrow$ What does this do?

Modifies the photon's amplitude for propagating between two points

↳ Look at the diagram : These virtual photons (all photons are virtual to some extent)

By that all I mean is they couple to fermion lines on the ends.

 $: -iQ|e_0|\gamma^\mu$

↳ Suppose I bundle up the $iQ|e_0|$ part w/ the photon propagator

Taking $Q=1$, we have,

$$\tilde{D}_{\mu\nu}(q) = \frac{-ie_0^2 g_{\mu\nu}}{q^2[1-(\tilde{\Pi}(q)-\tilde{\Pi}(0))]} = \frac{-ig_{\mu\nu}}{q^2}\underbrace{\left[\frac{e_0^2}{1-(\tilde{\Pi}(q)-\tilde{\Pi}(0))}\right]}_{e(q)\ :\ \text{Momentum dependent electric charge}}$$

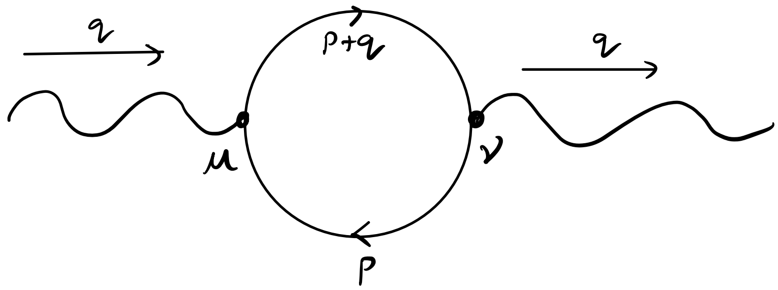
- The real life electric charge, renormalized by interactions w/ vacuum is given by,

$$|e| = \frac{|e_0|}{[1-(\tilde{\Pi}(q)-\tilde{\Pi}(0))]}$$

$$\Downarrow$$

This is telling us that e actually depends on the momentum transferred by the virtual photon, (the mediator of EM force)

- [SCREENING] : An explanation for this feature of the dielectric vacuum can be provided by concept of 'screening'.



In what follows we will require an expression for the one-loop amplitude in Fig. 41.5, which may be translated from diagram to equation, giving

$$\mathrm{i}\tilde{\pi}(q) = (-1)(-\mathrm{i}e_0)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu \frac{\mathrm{i}}{\not{p} - m} \gamma^\nu \frac{\mathrm{i}}{\not{p} + \not{q} - m}\right). \qquad (41.15)$$

This integral can be done. With the inclusion of the counterterm it is found that $\tilde{\pi}(q) - \tilde{\pi}(0)$, the one-loop contribution to $\tilde{\Pi}(q)$, is given by

$$\tilde{\pi}(q) - \tilde{\pi}(0) = -\frac{e_0^2}{2\pi^2} \int_0^1 \mathrm{d}x\, (x - x^2) \ln\left(\frac{m^2}{m^2 - (x - x^2) q^2}\right). \qquad (41.16)$$

Now let us tie everything to see this "Screening"

- To the second order, the renormalized propagator is given by

$$\begin{aligned} \tilde{D}_{\mu\nu}(q) &= \frac{-\mathrm{i}g_{\mu\nu}}{q^2} \frac{e_0^2}{\{1 - [\tilde{\pi}(q) - \tilde{\pi}(0)]\}} \\ &\approx \frac{-\mathrm{i}g_{\mu\nu} e_0^2}{q^2} \left[1 + \tilde{\pi}(q) - \tilde{\pi}(0)\right]. \end{aligned}$$

$(q^2 = -\vec{q}^{\,2})$

- To see the consequence of this on electrostatics, then we need static version of the propagation, obtained by

$$\tilde{\pi}(-\boldsymbol{q}) - \tilde{\pi}(0) = -\frac{e_0^2}{2\pi^2} \int_0^1 \mathrm{d}x\, (x - x^2) \ln\left(\frac{m^2}{m^2 + (x - x^2)\boldsymbol{q}^2}\right)$$

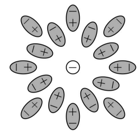
- Expanding this in the limit $q^2 \ll m$ $\quad \tilde{D}_{\mu\nu}(\boldsymbol{q}) \approx \frac{\mathrm{i}g_{\mu\nu} e_0^2}{\boldsymbol{q}^2}\left(1 + \frac{\alpha}{15\pi} \frac{\boldsymbol{q}^2}{m^2},\right)$, $\alpha = \frac{e_0^2}{4\pi}$

A very important interpretation to remember : is the Born approximation interpretation of the momentum space propagator as proportional to a matrix element of the potential $V(\vec{r})$ via

$$\tilde{D}_{00}(\vec{q}) \propto \int d^3 r \; e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r}) \xrightarrow[\text{transform of } \tilde{D}_{00}(\vec{q})]{\text{Inverse Fourier}} V(\boldsymbol{r}) = -\left\{\frac{\alpha}{|\boldsymbol{r}|} + \frac{4\alpha^2}{15m^2}\delta^{(3)}(\boldsymbol{r})\right\}.$$

- First term is Coulomb's potential for a point charge
- Second term is the correction due to the screening of the electron charge

- This screening effect can be visualized by the following diagram, where the pairs represent effective dipoles.
- This screening is known as **vacuum polarization**



- **The most important thing is that this can be measured!**
  - Lamb shift
    - Shift in the Hydrogen energy levels in the 2S(1/2) to 2P(1/2) of -27 MHz
    - This is only a small part of the Lamb shift (+1057 MHz)
      - The main part of the Lamb shift can be seen from the corrections to the electron propagator

Renormalization group and electric charge

$$\beta = \mu \frac{d|e|}{d\mu}$$

Using our equation for the renormalized electric charge we can expand for small $\tilde{\pi}(q)$ to obtain

$$e^2(\mu) = \frac{e_0^2}{1-(\tilde{\pi}(\mu)-\tilde{\pi}(0))} \approx e_0^2[1+(\tilde{\pi}(\mu)-\tilde{\pi}(0))] + O(e^4), \tag{41.21}$$

or

$$|e| \approx |e_0|\left[1+\frac{1}{2}(\tilde{\pi}(\mu)-\tilde{\pi}(0))\right]. \tag{41.22}$$

The β-function is then given by

$$\beta = \mu\frac{\mathrm{d}|e|}{\mathrm{d}\mu} = \frac{1}{2}\mu|e_0|\frac{\mathrm{d}\tilde{\pi}(\mu)}{\mathrm{d}\mu}. \tag{41.23}$$

Now we use eqn 41.16 which states that

$$\tilde{\pi}(\mu)-\tilde{\pi}(0) = -\frac{e_0^2}{2\pi^2}\int_0^1 \mathrm{d}x\,(x-x^2)\ln\left[\frac{m}{m^2-(x-x^2)\mu^2}\right]. \tag{41.24}$$

In the large energy scale limit ($\mu \gg m$) differentiation gets us

$$\frac{\mathrm{d}\tilde{\pi}(\mu)}{\mathrm{d}\mu} = \frac{e_0^2}{\pi^2\mu}\int_0^1 \mathrm{d}x\,(x-x^2) = \frac{e_0^2}{6\pi^2\mu}. \tag{41.25}$$

Note that, to the order to which we're working, it's permissible to swap e_0 for e on the right-hand side in what follows.

- We want to see how the electric charge changes depending on the size of the momentum that the photon is carrying!
- We will use the beta-function here given by

$$\Rightarrow \boxed{\beta = \frac{d|e|}{d\mu} = +\frac{|e|^3}{12\pi^2}}$$

- The positive sign is the most important part!
- This means that the electromagnetic coupling is increasing with increasing energy scale

- This also means, the coupling is increasing with decreasing length scale
- This means, as you go closer to the electron, the it seems to be more charged!
  - Makes sense with the screening interpretation : Getting closer to the electron you penetrate the

cloud of screening and the charge seems to increase!

We have a measurable prediction from the theory. Interestingly this calculation can also be done in non-relativistic quantum mechanics using Born's approximation, which says that the amplitude for scattering from a potential $V(\boldsymbol{r})$ is $\langle p'|\mathrm{i}\hat{T}|p\rangle = -\mathrm{i}\tilde{V}(\boldsymbol{q})(2\pi)\delta(E_{\boldsymbol{p}'} - E_{\boldsymbol{p}})$, where $\tilde{V}(\boldsymbol{q}) = \langle \boldsymbol{p}'|\hat{V}(\boldsymbol{r})|\boldsymbol{p}\rangle = \int \mathrm{d}^3r\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{r}} V(\boldsymbol{r})$. Comparing with our quantum field theory result, we see that Born says that our scattering potential is

$$\tilde{V}(\boldsymbol{q}) = \frac{-g^2}{|\boldsymbol{q}|^2 + \mu^2}. \tag{20.25}$$

What function has a Fourier transform that looks like this? The answer is

$$V(\boldsymbol{r}) = -\frac{g^2}{4\pi|\boldsymbol{r}|}\mathrm{e}^{-\mu|\boldsymbol{r}|}; \tag{20.26}$$

the potential dies away over a distance $1/\mu$. But this is just Yukawa's potential![8] This is the reason why the $\psi^\dagger\psi\phi$ theory is often called Yukawa theory.