2 Renormalization : The problem and the solution

Tutorial notes made by using Chapter 32 of Lancaster and Blundell's "QFT for the Gifted Amatuer"

> *It turns out that the obvious generalization of this idiotically simple manipulation gets rid of all of the infinities for any field theory with polynomial interactions, to any order in perturbation theory.*
> Sidney Coleman (1937–2007)

- In the last section we saw that, these dressed particles are different than the ones we have dealt with from the beginning while canonically quantizing our theories.
    - This further means that the perturbation theory we developed is at risk! Why? Because we did everything using the bare parameters m and λ, whereas in reality when we actually do measure things, it won't be these parameters but the dressed ones.
- The solution to this problem lies in making a shift to our parameters.
    - As a bonus, this also solves the problem of divergent amplitudes.

We divide our discussion into two parts:

1. Where do these divergences come from?
2. How an "idiotically simple" mathematical workaround actually fixes this problem.

2.1 Problem of divergences

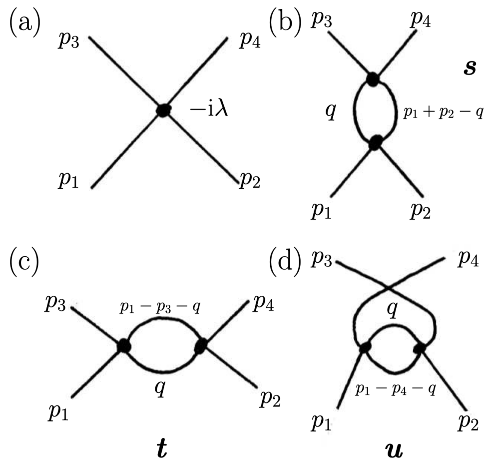
Let's examine three simple integrals (why these three? Cause these kind of integrals show up in computing actual amplitudes). For a real, finite and positive,

$$\begin{aligned}
\int_a^\infty \mathrm{d}x\, x^n &= \left[\frac{x^{n+1}}{n+1}\right]_a^\infty \text{ diverges for } \; n \geq 0, \\
\int_a^\infty \frac{\mathrm{d}x}{x} &= [\ln x]_a^\infty \text{ diverges}, \\
\int_a^\infty \frac{\mathrm{d}x}{x^m} &= \left[\frac{x^{-m+1}}{-m+1}\right]_a^\infty = \frac{a^{-m+1}}{m-1} \text{ for } m > 1.
\end{aligned}$$

Let's return to our Lagrangian for ϕ^4,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

Upto second order in perturbation theory we have the following diagrams for two -particle scattering (The amplitude for two-particle scattering, which is equal to the sum of all connected, amputated Feynman diagrams with four external legs),



Let's see how this unfolds. We've encountered the first diagram before. The result is

$$\mathrm{i}\mathcal{M}_{\mathrm{a}} = -\mathrm{i}\lambda.$$

For the subsequent diagrams, the integral we'll need is

$$\int_0^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{(p-q)^2 - m^2 + \mathrm{i}\epsilon} = -4\mathrm{i}a \ln\left(\frac{\Lambda}{p}\right),$$

where a is some numerical constant whose exact value isn't important to us. Notice that the limits of this integral are zero and Λ. The parameter Λ is a large momentum cut-off, and we want to send $\Lambda \to \infty$. By counting powers we see that there are four powers of momentum on top and four below, telling us that the integral $\sim \int \frac{\mathrm{d}^4 q}{q^4}$ is logarithmically divergent. In fact, for the diagrams in Figs. 32.1(b)–(d) we obtain the amplitudes:

$$\mathrm{i}\mathcal{M}_{\mathrm{b}} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln\left[(p_1 + p_2)^2\right]\right\} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln s\right\},$$

$$\mathrm{i}\mathcal{M}_{\mathrm{c}} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln\left[(p_1 - p_3)^2\right]\right\} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln t\right\},$$

$$\mathrm{i}\mathcal{M}_{\mathrm{d}} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln\left[(p_1 - p_4)^2\right]\right\} = \mathrm{i}a\lambda^2 \left\{\ln \Lambda^2 - \ln u\right\}.$$

All of these tend to ∞ as $\Lambda \to \infty$. The total amplitude for two-particle scattering is given by the sum of these diagrams, giving

$$\begin{aligned} \mathrm{i}\mathcal{M} = \quad &= \mathrm{i}\left(\mathcal{M}_{\mathrm{a}} + \mathcal{M}_{\mathrm{b}} + \mathcal{M}_{\mathrm{c}} + \mathcal{M}_{\mathrm{d}}\right) \\ &= -\mathrm{i}\lambda + \mathrm{i}a\lambda^2 \left\{3\ln\Lambda^2 - \ln s - \ln t - \ln u\right\}, \end{aligned}$$

and so we need to find a way to tame the $\mathrm{i}a\lambda^2 \left\{3\ln\Lambda^2\right\} \propto \ln\Lambda$ term.

- The divergence can be avoided if we choose a large finite cutoff Λ
    - Recall what this means : We are deliberately avoiding any fine-details below a length scale $\frac{1}{\Lambda}$ - Frequently done in condensed matter physics where the smallest length scale is set usually by atomic sizes / spacings/ etc
- For fundamental particles, there is no reason for a scale like this to exist. To make more sense of how theories/parameters change when we change energies, we will discuss Renormalization Group eventually!
- What if we keep $\Lambda \neq \infty$? Introducing a finite cutoff has a tragic consequence : Our amplitudes start becoming scale dependent (This statement should make more sense considering what we covered in tutorial 0)
- One strategy to avoid this is to add terms to the Lagrangian that, when we do the perturbation expansion up to some order (second order say), will remove the dependence of amplitudes on Λ.
    - Although we should still expect the higher order terms to diverge (i.e. blow up as $\Lambda \to \infty$), we'll at least have healed the theory to the extent that we can obtain a prediction of physical behaviour that is valid up to second order

2.2 COUNTERTERMS UNITE!

- We will add terms to the Lagrangian to get rid of this Λ dependence,
    - These terms are called **Counterterms**
- For ϕ^4 at second order, the divergent part of the amplitude looked like

$$6ia\lambda^2 \ln \Lambda \tag{1}$$

So lets change the Lagrangian by adding the term?

$$\mathcal{L} \to \mathcal{L} - \left(\frac{6a\lambda^2 \ln \Lambda}{4!}\right)\phi^4. \tag{2}$$

Now our Lagrangian looks like,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{C^{(2)}}{4!}\phi^4, \tag{3}$$

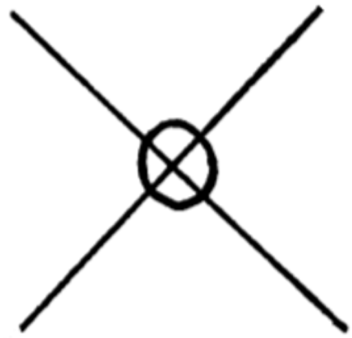
where $C^{(2)} = -6a\lambda^2 \ln \Lambda$ (The (2) denotes that we are getting rid of divergences at second order)

- Let us use this Lagrangian now and recall our canonical quantization! The Dyson expansion of the S-operator is,

$$\hat{S} = T\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x \frac{1}{4!}\left(\lambda\hat{\phi}^4 - C^{(2)}\hat{\phi}^4\right)}. \tag{4}$$

- Every order of the expansion therefore also includes a contribution from the counterterm, whose Feynman diagram is shown in fig below.

  and has the same behaviour as the $\frac{\lambda}{4!}\phi^4$ term.
    - It represents a vertex, instead of $-i\lambda$, it has $iC^{(2)}$



- We calculate amplitudes to second order in λ and (we only need to) include one diagram involving a counterterm and we obtain

$$i\mathcal{M}^{(2)} = -i\lambda + ia\lambda^2 \left[3\ln\Lambda^2 - \ln s - \ln t - \ln u\right] + iC^{(2)}, \tag{5}$$

and substituting $C^{(2)} = -6a\lambda^2 \ln\Lambda$ yields

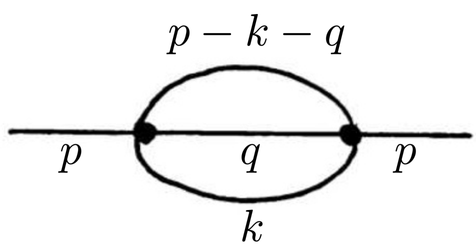
$$i\mathcal{M}^{(2)} = -i\lambda - ia\lambda^2 (\ln s + \ln t + \ln u).$$

- This amplitude depends on the momenta of the incoming and outgoing particles (which is no problem) but crucially it doesn't depend on Λ.
- We've done what we set out to do! Predictions for the theory will now not depend on an arbitrary cut-off Λ, and hence won't yield infinity when $\Lambda \to \infty$

2.3 Taming an integral

- Our plan :
  1. Start calculating the diagram using the original Lagrangian.
  2. See a divergence? Add counterterm
  3. Work out the coefficient in front of the counterterm
- What if you add a counterterm to cancel the second order and then you have infinities showing up in third order? Then you fix that, then you need to go to fourth ..,
  - Such a theory, where you need infinite amount of counterterms $\to$ non renormalizable
- On the other hand, if you need a finite amount of counterterms (like three for QED), those theories are renormalizable.

An example : Saturn Diagram



The amplitude for this diagram is given by,

$$\mathrm{i}\mathcal{M} = \frac{(-\mathrm{i}\lambda)^2}{6} \underbrace{\int_0^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{(p-q-k)^2 - m^2 + \mathrm{i}\epsilon}}_{I}. \tag{1}$$

- ▶ This is getting out of hand, and this is not THAT complicated of a diagram either. Let's come up with a system:
  - Write the integral as a polynomial. How can we do that? *Feynman diagrams can be expanded as Taylor series in the external momentum* (in this case, p)
  - The polynomial will represent the diagram. There will be divergent coefficients, that will reflect the divergence of the integral.
  - We pick counterterms to cancel these divergences in these coefficients.

Let the integral part of eq.(1) as a series around $p = 0$ given by,

$$I = \alpha + \tilde{\alpha} p + \beta p^2 + \tilde{\beta} p^3 + \gamma p^4 + \ldots, \tag{2}$$

where the tilde terms can be taken away because of $\phi = -\phi$ symmetry in the original Lagrangian, leaving us with,

$$I = \alpha + \beta p^2 + \gamma p^4 + \ldots, \tag{3}$$

Using this, how do we start to find the counterterms?

1. The diagram is quadratically divergent:
   - 8 powers of momentum on top, 6 on bottom
   - Setting external momentum to zero, $p = 0$, we have

     $$I = \alpha \tag{4}$$

     telling us that α is quadratically divergent.

   We need **one counterterm** that **diverges quadraticaly**

2. Differentiate the Saturn integral twice wrt p, you get $\rightarrow$ log divergence. (As we reduced the momenta by two)
   - Set $p = 0$ after this, you will get,

     $$I'' = 2\beta \tag{5}$$

     telling us that β diverges logarithmically.

   We need **another counterterm** which is log dependent on the cutoff. This will have a Feynman rule that it appears multiplied by p^2.

3. If we differentiate the integral two more times (reaching gamma), we reach a convergent integral, so we don't need any more counterterms.

ADD in the COUNTERTERMS

- ▶ So we need **two counterterms** to cancel the divergences in our Saturn integral I
  1. One with coefficient A quadratically dependent on Λ

2. One with coefficient B log dependent on Λ

- We also need to ensure that the B counterterm $\rightarrow$ gives a Feynman amplitude that has a p^2 dependence. Simultaneously, the A counterterm needs to have no p dependence.
  - We get this by,

$$\mathcal{L}_{\text{ct}} = \frac{A}{2}\phi^2 + \frac{B}{2}(\partial_\mu \phi)^2, \tag{6}$$

  - Adding the counterterms we have,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{A}{2}\phi^2 + \frac{B}{2}(\partial_\mu \phi)^2 + \frac{C}{4!}\phi^4. \tag{7}$$

  Note carefully that all these counterterms have the same form as the terms in the original Lagrangian, just different coefficients.

Feynman rules for renormalized ϕ^4 theory

- A factor $\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$ for each propagator.
- A factor $-\mathrm{i}\lambda$ for each interaction.
- Add sufficient counterterm diagrams to cancel all infinities.
- A factor $\mathrm{i}\left(B^{(n)}p^2 + A^{(n)}\right)$ for each counterterm propagator, where n is the order of the diagram.
- A factor $\mathrm{i}C^{(n)}$ for each interaction counterterm, where n is the order of the diagram.
- All other rules regarding integrating, symmetry factors and overall energy momentum conserving delta functions are identical as for the case of unrenormalized perturbation theory.

Why ϕ^2 in front of A?

There were a lot of questions in the class with this. The answer is that, we are finding the corrections to the simple two point propagator here. Look at the diagram, its a $1 \rightarrow 1$ scattering (whereas before the diagrams were talking about $2 \rightarrow 2$ scattering and hence the counterterm went in front of the ϕ^4 interaction term.

How did we get all the counterterms we needed from just these diagrams?

Yes, there are many more diagrams of course. The fact that we only need these diagrams will need a more rigorous arguments (ideally a proof) which I think you can find in Peskin and Schroeder. The point of this exercise was, "How to find counterterms" for a diagram or set of diagrams (we choose the diagrams conveniently such that the first 4 diagrams gave us the correction to the interaction vertex and the saturn diagram gave correction needed to the massive propagator)

2.4 Interpretation of counterterms

- The counterterms shift our parameters from fictional to real.
- AND, they simultaneously get rid of infinities which forces our theory to describe real life.
- Lets start with the unrenromalized Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1}$$

where m and λ are the same constants as we know.

  - Turning on coupling $\rightarrow$ gives infinities $\rightarrow$ unless we cut off the integrals at some momentum Λ
  - Step 1 : To fix this divergence and work with real parameters $\rightarrow$ we accept that we are dealing with *dressed particles* (This does not by itself remove the infinities)
  - Step 2 : Add in the counterterms! This is what removes infinities.

$$\begin{aligned} \mathcal{L}' = \frac{1}{2}(\partial_\mu \phi_{\mathrm{r}})^2 - \frac{m_{\mathrm{P}}^2}{2}\phi_{\mathrm{r}}^2 - \frac{\lambda_{\mathrm{P}}}{4!}\phi_{\mathrm{r}}^4 \\ + \frac{B}{2}(\partial_\mu \phi_{\mathrm{r}})^2 + \frac{A}{2}\phi_{\mathrm{r}}^2 + \frac{C}{4!}\phi_{\mathrm{r}}^4, \end{aligned} \tag{2}$$

    where we have rescaled the field using, $\phi = \sqrt{Z}\phi_r(x)$.

    - ϕ_r is the renormalized field
    - Z is independent of three momentum here!

    This theory with ϕ_r, gives us sensible non-infinite results for $\Lambda \rightarrow \infty$

Lets check this example out to illustrate what we mean, To see the consequence of renormalization, we now collect the coefficients of the renormalized Lagrangian

$$\mathcal{L} = \frac{1+B}{2}(\partial_\mu \phi_{\rm r})^2 - \frac{(m_{\rm P}^2 - A)}{2}\phi_{\rm r}^2 - \frac{(\lambda_{\rm P} - C)}{4!}\phi_{\rm r}^4,$$

which suggests that the counterterms represent shifts in the parameters in the Lagrangian. Writing $-A = \delta m^2$, $-C = \delta\lambda$ and $B = \delta Z$ we have

$$\mathcal{L} = \frac{1+\delta Z}{2Z}(\partial_\mu \phi)^2 - \frac{(m_{\rm P}^2 + \delta m^2)}{2Z}\phi^2 - \frac{(\lambda_{\rm P} + \delta\lambda)}{4!Z^2}\phi^4,$$

where we've restored the original, unrenormalized fields. From this, we read off that we can relate our original and renormalized Lagrangians through

$$\phi = \sqrt{Z}\phi_{\rm r}, \quad Z = 1 + \delta Z, \quad m^2 = \frac{(m_{\rm P}^2 + \delta m^2)}{Z}, \quad \lambda = \frac{(\lambda_{\rm P} + \delta\lambda)}{Z^2}.$$

This tells us that renormalization is simply an exercise in shifting parameters.

To truly see what's going on it may make more sense now to replay the argument backwards. We start with the unrenormalized Lagrangian and shift from parameters m and λ to parameters $m_{\rm P}$ and $\lambda_{\rm P}$. We start by renormalizing the fields to obtain

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \\ &= \frac{Z}{2}(\partial_\mu \phi_{\rm r})^2 - \frac{Zm^2}{2}\phi_{\rm r}^2 - \frac{Z^2\lambda}{4!}\phi_{\rm r}^4.\end{aligned}$$

We then use the shifts

$$Z = 1 + \delta Z, \quad Zm^2 = m_{\rm P}^2 + \delta m^2, \quad Z^2\lambda = \lambda_{\rm P} + \delta\lambda,$$

giving us

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_{\rm r})^2 - \frac{m_{\rm P}^2}{2}\phi_{\rm r}^2 - \frac{\lambda_{\rm P}}{4!}\phi_{\rm r}^4 + \frac{\delta_Z}{2}(\partial_\mu \phi_{\rm r})^2 - \frac{\delta_m^2}{2}\phi_{\rm r}^2 - \frac{\delta_\lambda}{4!}\phi_{\rm r}^4,$$

allowing us to identify the counterterms A, B and C as the shifts in the parameters $-\delta m^2$, δZ and $-\delta\lambda$.

2.5 Conclusion

- (Recall how we started QFT) : We started with Lagrangians in QFTI assuming masses m and coupling constant λ
- (We were doing it wrong) : Those were the wrong masses and coupling constants. And further, as we used perturbation theory, we were doing it by expanding our theory around the wrong coupling constant!
- (Solution is counterterms) : The natural argument is that : By taking the CTs we are making the shifts needed and getting rid of infinities. The real argument on the other hand is: by making these shifts we are representing our theory with the correct parameters (which it deserves). An "bonus" if you may is that it gets rid of infinites. In my books, it had to get rid of these infinites or else we probably were still missing something physical. *In reality, we are not adding counterterms at all, we are just giving the theory the real parameters it deserves.*

- ► (What is RN?) : *Renormalization is not, therefore, an exercise in hiding infinities, it's an exercise in making a theory describe real life.*