Notation

- Bold letters : $\boldsymbol{p}, \boldsymbol{x}$ are 3 vectors
- Non bold (usually latin) letters : p, x are 4 vectors

1 Field Strength Renormlization

Goal of this section

- Investigate the analytic structure of two-point correlation function

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \quad \text{or} \quad \langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle.$$

- Compare with free field theory, where

$$\langle 0|\, T\phi(x)\phi(y)\, |0\rangle$$

has the simple interpretation - *Amplitude for a particle to propagate from* $y \to x$

Checkpoint 1 colframe
To what extent is this true for interacting theory too?

Assumptions?

- Just general principles of QM and relativity
- No interacting model assumed
- Stick to scalar fields (but all of these arguments can be extrapolated to correlation functions of fields with spins)

1.1 Build Hilbert space

$$\hat{H}, \quad \hat{\boldsymbol{P}} \tag{1}$$

For general interacting systems, these don't commute so we try to break down different cases (by thinking what type of states are we including for eigenstates of $\boldsymbol{P}$)

1. Single free particle - Well defined energy and momentum
2. Bound state from a system of particle - Well defined energy and momentum. Difference from single particle case is that that it will have parameters like binding energy. The mass is well defined too : Sum of individual masses minus binding energy

3. Unbound system of particles - Everything same as (2), but the mass is not well defined here. Why? Because the energy can vary indefinitely of momentum - you go to the rest frame the system's total momentum is zero, but the energy can be anything depending on how fast these particles are moving.

This will be the basis of all our *particle states* that can exist.

Start building

Let $|\lambda_0\rangle$ as the zero-momentum states,

$$P\,|\lambda_0\rangle = 0 \tag{2}$$

of these three types. The energy of these states is,

$$H\,|\lambda_0\rangle = E_0(\lambda)\,|\lambda_0\rangle \tag{3}$$

The λ takes care of all other parameters. For a scalar field, it will run over all possible masses (you can add spin tot he mix if you are dealing with other fields).

Any state of the theory can be taken by making a boost to arbitrary momentum $\boldsymbol{p}$,

$$|\lambda_0\rangle \xrightarrow[\boldsymbol{p}]{\text{boost}} |\lambda_{\boldsymbol{p}}\rangle \tag{4}$$

Then, from Lorentz invariance, the boosted states are also eigenstates of the Hamiltonian,

$$H\,|\lambda_{\boldsymbol{p}}\rangle = E_p(\lambda)\,|\lambda_{\boldsymbol{p}}\rangle \tag{5}$$

How are the eigenvalues of energy and momentum related?

By the following famous dispersion relation,

$$E_p(\lambda) \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2} \tag{6}$$

This is how mass is defined in a relativistic theory and works really well for the first two cases (single particle and bounded particle). Sitting in the rest frame, the single particle case just gives energy equals mass. The bound state gives sum of all masses minus binding energy.

But in the third case, m_λ is defining the energy of this system in the rest frame (not something we usually call mass!?) because as opposed to the first two cases where it is a number, in this case it is a *continuous function* of the *individual momenta* (not total momenta), their positions, strength of interaction. A complicated expression, which in the bound state is encapsulated in the binding energy.

The following plot,

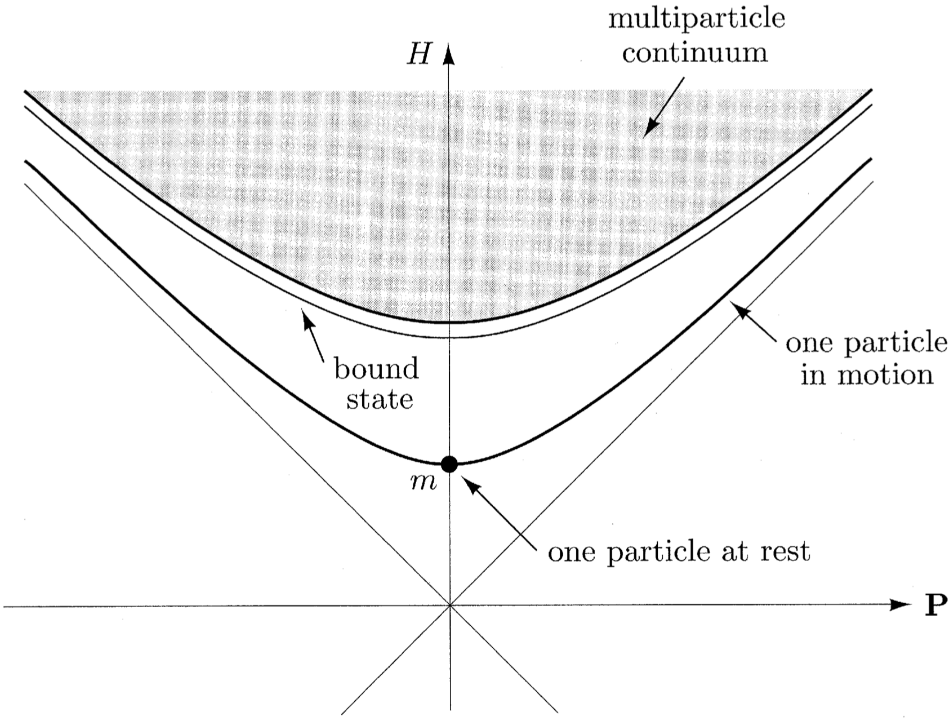

Figure 7.1. The eigenvalues of the 4-momentum operator $P^\mu = (H, \mathbf{P})$ occupy a set of hyperboloids in energy-momentum space. For a typical theory the states consist of one or more particles of mass m. Thus there is a hyperboloid of one-particle states and a continuum of hyperboloids of two-particle states, three-particle states, and so on. There may also be one or more bound-state hyperboloids below the threshold for creation of two free particles.

Figure 1:

Why are these considerations important? If we want to write down a completeness relation for our Hilbert space, we need to consider all these scenarios.

The completeness relation is given by,

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum\!\!\!\!\!\!\!\int_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}(\lambda)}\,|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\,, \tag{7}$$

The first term is the interacting vacuum and the part after $\sum\!\!\!\!\!\!\int$ is just the one particle state completeness relation we know.

The sumint is needed because of what we explained in the diagram : For adding individual particles in the mix or bound states, sometimes we will take a sum or an intergral based on which part of the y-axis are we working on.

1.2 Analyzing the Two point function

We are looking at vacuum to vacuum transition with these operators in the middle (assume $x^0 > y^0$,

$$
\begin{aligned}
\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle &\xrightarrow{x^0>y^0} \langle\Omega| \phi(x)\mathbf{1}\phi(y) |\Omega\rangle &(1)\\
&= \langle\Omega| \phi(x) \left(|\Omega\rangle \langle\Omega|\right) \phi(y)\ \ |\Omega\rangle &(2)\\
&\quad + \langle\Omega| \phi(x) \left(\sum\!\!\!\!\!\!\int_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}(\lambda)} |\lambda_{\mathbf{p}}\rangle \langle\lambda_{\mathbf{p}}| \right) \phi(y) |\Omega\rangle \\
&= \langle\Omega| \phi(x) |\Omega\rangle \langle\Omega| \phi(y) |\Omega\rangle &(3)\\
&\quad + \sum\!\!\!\!\!\!\int_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}(\lambda)} \langle\Omega| \phi(x) |\lambda_{\mathbf{p}}\rangle \langle\lambda_{\mathbf{p}}| \phi(y) |\Omega\rangle \\
& &(4)
\end{aligned}
$$

Note : Just keep in find that all the fields in there are operators.

Deal with the first term

The first term is a VEV and these are usually zero. If they are not zero in say spontaneous symmetry breaking theory, then you move to a new field which actually has a zero VEV and you deal with the consequences of that with Higgs Boson or other goldstone bosons. So when dealing with such things, we are already assuming that we are in the broken phase. Right now, we just assume that the first term is zero and move on.

Deal with the second term

In order to do anything more with the second term, we first need to figure out the brakets inside the term. Start by dissecting the first braket,

$$
\begin{aligned}
\left\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\right\rangle &= \left\langle\Omega \left| e^{iP\cdot x}\phi(0)e^{-iP\cdot x} \right| \lambda_{\mathbf{p}}\right\rangle \\
&= \left\langle\Omega|\phi(0)|\lambda_{\mathbf{p}}\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_{\mathbf{p}}} \\
&= \left\langle\Omega|\phi(0)|\lambda_0\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_{\mathbf{p}}}
\end{aligned} \tag{5}
$$

The following steps are followed in the lines,

1. Translated the $x \to 0$ using momentum as the translation operator
2. Vacuum has momentum zero, so $\langle\Omega| e^{iP\cdot x} = \langle\Omega| e^0 = |\Omega\rangle$ and The same is done on the second exponential, but here the operator is changed with it's eigenvalue.
3. We want to make everything in the rest frame of $|\lambda_{\boldsymbol{p}}\rangle$, so boost back in that rest frame. Where we will have some unitary operator doing the dirty work for us, $U|\lambda_{\boldsymbol{p}}\rangle = |\lambda_0\rangle$. Technically this is done in the following manner,

$$\underbrace{\langle\Omega| U^{-1}}_{1.\langle\Omega|}\underbrace{U\phi(0)U^{-1}}_{2.\phi(0)}\underbrace{U|\lambda_{\boldsymbol{p}}\rangle}_{3.|\lambda_0\rangle} e^{ipx}\Big|_{p^0=E_{\boldsymbol{p}}} \tag{6}$$

where we used,

   1. Vacuum is invariant under translations
   2. Scalar field doesn't transform under translation
   3. By Definition

The point 3.2 if very trivial for a Scalar field. But, if you check chapter 10 of Weinberg (Vol I/II don't recall), you will see that he does it for a more generic field. As long as we know how it transforms under boosts, we can do what we just did here. In other scenarios, we would have some representation R in front of the operator, etc.

Combine the two terms

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum\!\!\!\!\!\!\int_{\lambda} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} \,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \; e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}e^{-iE_p(x_0-y_0)} \tag{7}$$

We made the spacetime separation because, we know that we can write the following three integral as a four integral (which has been often when you are computing two point funcitons to begin with say to find the Feynman propagator, etc),

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}e^{-iE_p(x_0-y_0)}}{2E_p(\lambda)} = \int_{x_0>y_0} \frac{d^4p}{(2\pi)^4}\underbrace{\frac{i}{p^2-m^{\lambda 2+i\epsilon}}}_{D_F(p,m_\lambda)} e^{-ip(x-y)} \tag{8}$$

Plugging this into the integral above,

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \sum\!\!\!\!\!\!\!\int_{\lambda} \underbrace{\int_{x_0>y_0} \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2 - m^{\lambda^2+i\epsilon}}}_{D_F(p,m_\lambda)} e^{-ip(x-y)}}_{D_F(x-y,m_\lambda)} |\langle\Omega|\,\phi(0)|\lambda_0\rangle|^2 \tag{9}$$

The other scenario, $y_0 > x_0$ gives us exactly the same thing with x and y swapped. Finally we get,

$$\boxed{\langle\Omega|\,T\{\phi(x)\phi(y)\}\,|\Omega\rangle = \sum\!\!\!\!\!\!\!\int_{\lambda} |\langle\Omega|\,\phi(0)|\Omega\rangle|^2\, D_f(x-y, m_\lambda)} \tag{10}$$

This is the important result that we wanted and it is a fairly interesting expression. Let's take a moment to analyze this expression a bite more further.

Suppose we are far away from the region where the λ becomes a continuous variable, i.e. we are in the part where we can produce single particle state and some bound state above it. In that region what we are getting is just a sum,

$$\langle\Omega|\,T\{\phi(x)\phi(y)\}\,|\Omega\rangle = \sum_{\lambda} |\langle\Omega|\,\phi(0)|\Omega\rangle|^2\, D_f(x-y, m_\lambda) \tag{11}$$

What we are getting is a sum of Feynman propagators for different values of m_λ, with a weight in front.

We are just using the simplest thing which is a single scalar field in the Lagrangian. But what we are still getting a sum of many poles, each of these Feynman propagators are producing a pole for different value of lambda. m_1 is contributing to the sum (as expected) by a propagator, but even m_2 which corresponds to a bound state is also a part of the same sum.

Diagramatically,

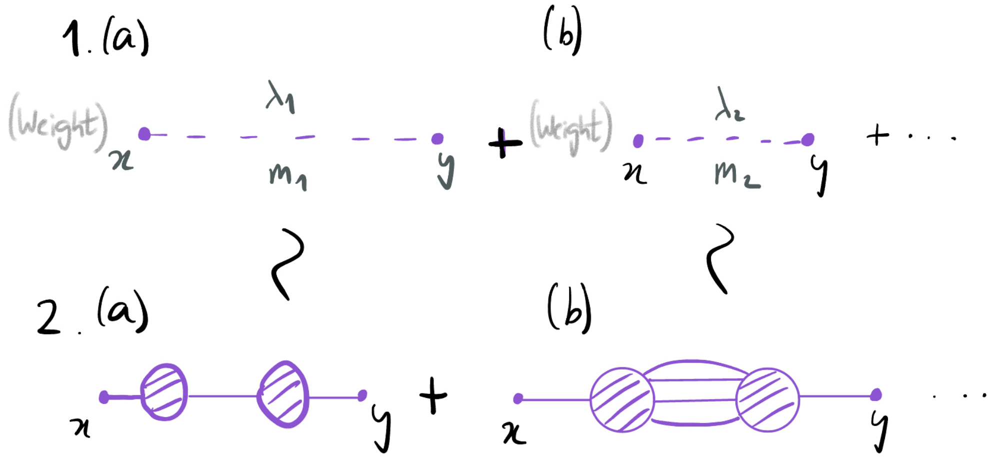

Figure 2:

We may think that the m_1 here is coming from the Lagrangian $\mathcal{L} = \frac{1}{2}m^2\phi^2$, but actually that is not guaranteed. If we traceback to where m_{λ_1} showed up for the first time - It is when we exchanged the 3-momentum integral with 4-momentum i.e. eq.(8). But it appeared there on the RHS because it was intrinsically hidden in the LHS denominator in the relativistic dispersion relation,

$$E_p(\lambda) = \sqrt{\boldsymbol{p}^2 + m_\lambda^2} \tag{12}$$

but this m_λ is not the same as the one defined in the lagrangian. It could indeed be the same number as the one in the lagrangian, or not. We do not know. What we know is that it is the **physical mass** of this on-shell particle. This is how m_λ showed up, both for the single particle and the bound state.

Note : In the diagram 2.(b) : whatever are the interactions between the propagators between the two blobs, they will produce poles in the Green's function, if it is a bound state. It of course gets more complicated when we have to also include the integral over λ and not only the sum.

Conclusion of this part

Even if we have only one mass parameter in the Lagrangian, we will still be producing a lot of poles. None of these masses, $m_1, m_2, m_3, \ldots$ are directly connected to that mass parameter in the Lagrangian.

This is one of the first instance one can see that, if you go to the more complete version of this theory, that relation between masses of particles at resonance that you actually see are not straightforwardly connected to the one in the Lagrangian.

1.3 Spectral representation

We can rewrite our main result eq.(10) in a new manner by introducing a new integral over M^2 (M has dimensions of mass). We of course do this by introducing a delta function,

$$
\begin{aligned}
\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle &= \int_0^\infty \frac{dM^2}{2\pi}(2\pi)\delta(M^2 - m_\lambda^2) \\
&\quad \sum\!\!\!\!\!\!\!\int |\langle\Omega| |\phi(0)|\lambda_0\rangle|^2 D_F(x-y, M^2) \qquad (1) \\
&= \int_0^\infty \frac{dM^2}{2\pi} \\
&\quad \underbrace{\sum\!\!\!\!\!\!\!\int (2\pi)\delta(M^2 - m_\lambda^2)\, |\langle\Omega| |\phi(0)|\lambda_0\rangle|^2}_{\rho(M^2)} D_F(x-y, M^2) \qquad (2)
\end{aligned}
$$

Giving us,

$$
\boxed{\rho(M^2) = \sum\!\!\!\!\!\!\!\int (2\pi)\delta(M^2 - m_\lambda^2)\, |\langle\Omega| \phi_0 |\lambda_0\rangle|^2} \tag{3}
$$

This is called as the *Kallen-Lehmann* spectral representation. Using this, the two point function becomes much more compact,

$$
\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2) D_F\left(x-y, M^2\right) \tag{4}
$$

What does this $\rho(M^2)$ tell us? It tells us about where the resonances are and where the continuum begins, which serves as a weight for the contributions of the propagators. We will see that we will be looking for poles when the sumint is a sum and branches when it is a continuum. Here is a quick plot ,

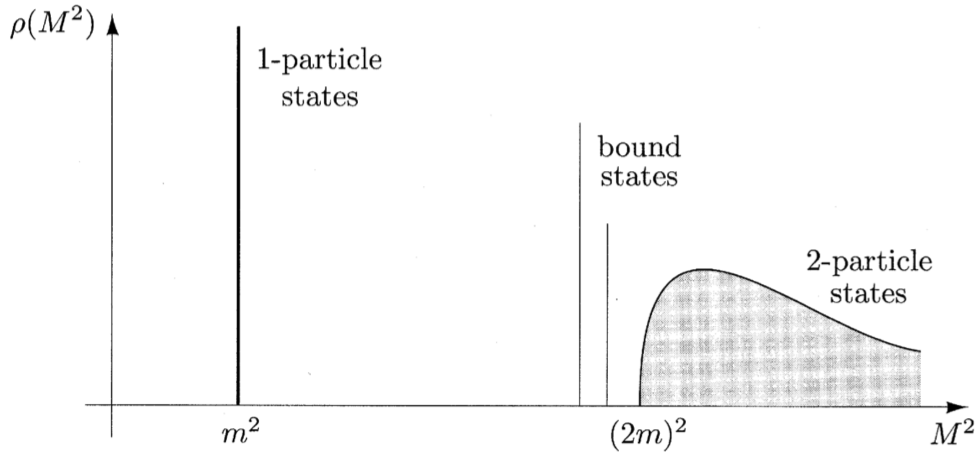

Figure 7.2. The spectral function $\rho(M^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

Figure 3:

Note that the one-particle states contribute an isolated delta function to the spectral density (i.e. for $m_\lambda = m_1 \equiv m$)

$$\rho\left(M^2\right) = 2\pi\delta\left(M^2 - m^2\right) \cdot \underbrace{\left|\langle\Omega|\,\phi(0)\,|1_0\rangle\right|^2}_{Z} + \left(\text{nothing else until} M^2 \gtrsim (2m)^2\right) \quad \textit{dotted} \tag{5}$$

Plugging this spectral density in the two point function, we have (in the approximation of one particle),

$$\langle\Omega|\, T\{\phi(x)\phi(y)\}\,|\Omega\rangle \approx \int_0^\infty dM^2 \delta(M^2 - m^2)\; Z\; D_F(x-y, M^2) \tag{6}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{iZ}{p^2 - m^2 + i\epsilon}\, e^{-ip(x-y)} \tag{7}$$

So, near the one particle pole of this theory, the two point function behaves in a very similar way to the free particle theory.

What is the difference between free particle and free theory?

- *Free theory*: It is the Lagrangian without any interactions. (??Just a gaussian on the fields??)
- *Free particle* : It is the free particle from a free theory. Such particles are the idealized FREE particle, meaning it has absolutely no interactions, and hence, nobody will ever even see them. Why? No interactions, no measurement.

The subtle things that comes now is that you could be talking about a *free particle of an interacting theory*, this is a particle that is just far away from any interactions that it could have, it is behaving as a free. Moving straight (if one tries to have an analogy with classical mech)

So what we have calculated is exactly this : it is the propagator of a free particle in an interacting theory. This motivates a lot of definitions,

- Z : Field strength renormalization.
- m : Physical mass (This is from the relativistic dispersion relation not the Lagrangian) - When you put a particle on shell, this is the mass it will have.
- m_0 : Bare mass (from the Lagrangian). Bare in the sense that this is the mass that would coincide with the mass for a free particle in a free theory. (For example this is the mass that an electron would have if you strip it out of all the interactions it could have).

Comment

An important thing to keep in mind is that, everything that we did here was non-perturbatively, and we reached something that is very closely related to a physical thing that we measure. This m has no relation to m_0 atleast a priori.

1.4 Reconciliation with Free theory

Suppose I make a simple (rescaling) change of variable in my Lagrangian,

$$\phi' = \frac{\phi}{\sqrt{Z}} \tag{1}$$

If I calculate the two point function as above, from scratch using this new field, I would just have another $\frac{1}{Z}$ factor in front of eq.(7).

$$\begin{aligned} \langle \Omega | \, T\{\phi'(x)\phi'(y)\} \, |\Omega\rangle &\approx \int_0^\infty dM^2 \delta(M^2 - m^2) \; D_F(x-y, M^2) && (2) \\ &= \int \frac{d^4p}{(2\pi)^4} \, \frac{iZ}{p^2 - m^2 + i\epsilon} \, e^{-ip(x-y)} && (3) \end{aligned}$$

By redefining my field (or renormalizing if I may), I got a new field that has near the one particle pole exactly the same analytical form as you had for propagator in the free theory because $m \neq m_0$ from the Lagrangian. This is what is done again and again and is renormalization.

Def: Renormalization

Trying to write the theory in terms of fields that better represent the physical state. T

Another crucial point to emphasize is that, we have not done a single loop calculation yet. We have not even assumed that our theory has a perturbative expansion where those loops will show up. This is an extremely general result, showing us that renormalization was needed even if there are no loops or infinities in the theory.

Away from the one particle pole

Most of the justification we made after defining the Kallen-Lehmann spectral representation was based on the assumption that we are having a one particle pole. We are now looking at what happens away from the pole,

$$\rho(M^2) \equiv \underbrace{2\pi\delta(M^2 - m^2)Z}_{\text{one particle}} + \sigma(m^2) \tag{4}$$

This is again represented in the fig.(3). The different heights of the delta functions coming from poles at single particle and bound state is due to different Z.

After you hit the $2m^2$, we get a function talking about multi-particle states. The bound states + multi-particle states is encapsulated by σ. The continuum part can be even further defined to be $\tilde{\sigma}$ (there could be further 3 particle bound states on top of this continuum, this can be quite complicated).

Let's just investigate the Fourier transform of the Greens function,

$$\int d^4x \, \langle\Omega| \, T \, \phi(x)\phi(0) \, |\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon} \tag{5}$$

$$= \frac{iZ}{p^2 - m^2 + i\epsilon} + (\text{Poles of Bound states}) + \int_{4m^2}^\infty dM^2\tilde{\sigma}(M^2 \tag{6}$$

The last term will form a branch cut (taking $\epsilon \to 0$), which will look like,

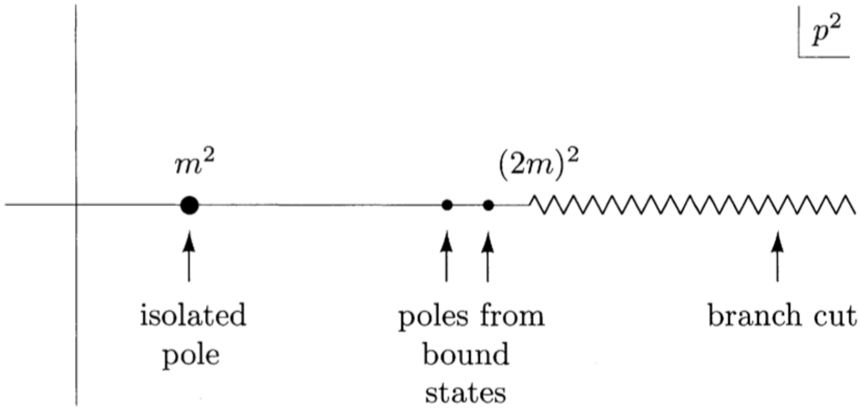

Figure 7.3. Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut, while bound states give additional poles.

Figure 4: The y-axis is the imaginary part of p^2 and x-axis is the real part of p^2

Now, if you compare this expression for the momentum space expression for the free field,

$$\int d^4x e^{ip\cdot x} \langle 0| T\phi(x)\phi(0) |0\rangle = \frac{i}{p^2 - m^2 + i\epsilon} \tag{7}$$

which tells us, when you go to free theory, you NEED,

$$Z \to 1 \tag{8}$$
$$m \to m_0 \tag{9}$$

telling us that all these bound states, multiparticle continuum is a result of the interactions (as we lose all these things in the free theory).

In Weinberg section 10.7, you can show that this spectral density has the positivity property, and

$$\int_0^\infty \rho(M^2) dM^2 = 1 \tag{10}$$

showing us that this spectral representation is indeed a weight telling us on how much this one particle, two particle and multiparticle states contribute. We can also show,

$$Z + \int_0^{inf} \sigma(M^2) dM^2 = 1 \tag{11}$$

which means that when Z goes to 1, $\sigma = 0$ and all the bound states disappear. On the contrary, when $Z = 0$, the very first single state disappear. This is the case when the theory is extremely strongly coupled. If you try to do the renormalization in that case, the ϕ' becomes a huge field which amount to integrating out this field (we will learn later what integrating out means soon). That is the case when the field in the Lagrangian does not produce a particle. This is when whatever particle has been produce has no field in the Lagrangian, this is a "composite" state (We will come back to that).This case of $Z = 0$ is a totally non perturbative and it is very difficult to calculate in that limit.