Plenary Tutorial 1,2 : Demystifying Renormalization

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> “Despite the experimental success of the theory...the fact that the infinities occur at all continues to produce grumbling...Dirac in particular always referred to renormalization as sweeping the infinities under the rug. I disagreed with Dirac and argued the point with him at conferences at Coral Gables and Lake Constance. Taking account of the difference between the bare charge and mass of the electron and their measured values is not merely a trick that is invented to get rid of infinities; it is something we would have to do even if everything was finite. There is nothing arbitrary or ad hoc about the procedure; it is simply a matter of correctly identifying what we are actually measuring.
>
> Steven Weinberg

1 Renormalization, quasiparticles and Fermi surfaces

Tutorial notes made by using Chapter 31 of Lancaster and Blundell's "QFT for the Gifted Amatuer"

- The transition from *non interacting* $\rightarrow$ *interacting* :
  - Not only lets particles scatter from each other
  - But also changes properties of particles like *mass and charge*
- We say that - these properties are **renormalized** in this transition. This process can be imagined as *particle **dressing**in interactions.*
  - Example 1 : Screening of a positive charge embedded in a metal.
    Positive charge will build a cloud of electron density around itself. Due to this, distant electrons will not feel it's full charge as the *nearby electron density* will *screen* the charge. Which will reduce it's *apparent value.*
  - Example 2 : Effective mass of an electron in a crystal
    The crystal contains immobile ion cores, which will interact electrostatically with the electron. Which means that it will gain an effective massm^*, different than that of a *bare electron in vacuum.*
- The difference between the above examples from *condensed matter physics* v/s *fundamental theory* like QED is that
  - Condensed matter : Can measure the mass of the electron both inside the material and in vacuo
  - Fundamental theory (QED) : Cannot remove the electrons from the vacuum of QED

1.1 Recap - Interacting and non-interacting theories

Non-interacting particles

- Simple example of non-interacting theory is the free field theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2 \tag{1}$$

Recap : Properties of non-interacting theory

- For a theory described by free Hamiltonian H_0,
  - we can define *freely evolving field operators* and *vacuum ground state*

$$\hat{H}_0|0\rangle = 0, \qquad \hat{\mathbf{p}}|0\rangle = 0, \qquad \langle 0|\hat{\phi}\left(x\right)|0\rangle = 0 \tag{2}$$

  - *Particles* : by definition are excitations of the vacuum
  - Creation/Annihilation operators create *single-particle* modes, one at a time. (Labeled by their momenta)
- Example : Create a relativistically normalized single particle

$$\langle p|\hat{\phi}^\dagger\left(x\right)|0\rangle = e^{ip\cdot x} \tag{3}$$

This is the amplitude of a field operator to *create* a *relativistically normalized* single particle with momentum p at position x

- Free Feynman Propagator

$$\tilde{G}_0(p) = \frac{i}{p^2 - m^2 + i\varepsilon} \tag{4}$$

  - This describes a *single particle* inserted into the system at y and removed at x.
  - For a *non-interacting theory*, this will not be affected by other particles in the system (Neither will it affect other particles in the system)
  - IMPORTANT : If we did not know the ***mass of the particle*** from before using $E_{\mathbf{p}}^2 - p^2 = m^2$, then we can read it off as the ***pole of the propagator/-Green's function.***

Interacting particles Consider a Hamiltonian as a *sum of the interacting and non interacting parts*

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}' \tag{5}$$

- $\lambda(T)$ starts small and approaches unity as we turn up "some parameter" T

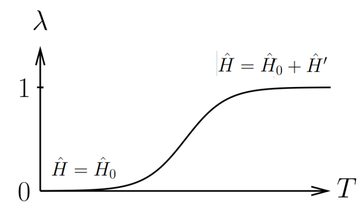


Fig. 31.1 Adiabatic turning on of $\hat{H}'$. The full Hamiltonian is given by $\hat{H} = \hat{H}_0 + \lambda(T)\hat{H}'$. This thought experiment allows us to examine what happens to particles in an interacting theory.

- As T grow, the whole system becomes cauldron of interactions (As T grows, the λ grows, which means that the interaction part of the Hamiltonian H' gets activated)
- These *new eigenstates* of the *full Hamiltonian* $\hat{H}$ are $|\mathbf{p}_\lambda\rangle$ (For the free Hamiltonian $\hat{H}_0$ they are $|\mathbf{p}\rangle$). This also affects the dispersion relation of the particles (Originally $E^2 = p^2 + m^2$)
- Putting a test particle in such a system is going to cause havoc. It will interact with all particles and anti particles pulling them out of the vacuum. The original particle is most likely to be lost. Do we have to abandon QFT which is after all based on the notion of creating and annihilating single particles? *Are there such things as* ***single particle excitation*** *in an interacting system?*
- They do indeed exist and are called as **dressed particles or quasi-particles.**

1.2 Quasiparticles

RENORMALIZATION

Quasiparticles are the excitations in interacting systems. They resemble free particles, but have different masses and interact differently with their environment. **The process of free particles turning into dressed particles is called renormalization.**

- Turn on interactions $\rightarrow$ Single non-interacting particle $\rightarrow$ Particle gets dressed by interactions and becomes Quasiparticle

$$\text{(Quasiparticle)} = \text{(Real particle)} + \text{(Interactions)} \tag{1}$$

Remark : It can be called as **dressed particle** or **renormalized particle** depending on context.

- After turning on interaction, the ground state of our theory is transformed

(i) $H|0\rangle = 0 \longrightarrow H|\Omega\rangle = 0$

Free vacuum   Interacting vacuum

(ii) $a^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle \longrightarrow a^\dagger_p|\Omega\rangle = \,??$

$\boxed{\text{☆ } a^\dagger|\Omega\rangle \text{ is not necessarily a single-particle excitation}}$

↳ $a^\dagger_p$ on $|\Omega\rangle$ is like a 'bull in a china shop'.

↳ Nothing stops it from making several excitations at once.

Q. Why are we looking at $a^\dagger_p$?

A. $\phi^\dagger(x) \propto \int_{\vec{p}} (a^\dagger_{\vec{p}} e^{ip.x} + a_p e^{-ip.x})$

- So in our interacting theory, we need to have a new operator to generate single particle excitations

↳ We define it as $q^\dagger_p \rightarrow q^\dagger_{\vec{p}}|\Omega\rangle = |\vec{p}_\lambda\rangle$ Indicating that we have interactions.

↳ We are clueless w.r.t what $q^\dagger_p$ looks like.

- Let us see what $a^\dagger_p|\Omega\rangle$ gives us:

$$a^\dagger_{\vec{p}}|\Omega\rangle = \underbrace{|\vec{p}_\lambda\rangle\langle\vec{p}_\lambda|\, a^\dagger_{\vec{p}}|\Omega\rangle}_{\text{One of the particle we might end up making.}} + \underbrace{\sum \text{(Multiparticle parts)}}_{\text{Other particles w. multiparticle states, whose momenta add up to } \vec{p}.}$$

In CMP, these can correspond to electron-hole pairs. $\Big\}$ e.g. 2 particle + 1 aparticle

$q^\dagger_{\vec{p}_1} q^\dagger_{\vec{p}_2} q_{\vec{q}_1+\vec{q}_2-\vec{p}}|\Omega\rangle$

- This is actually even more complicated

↳ We will only 'approximately' make a single particle state $|\vec{p}_\lambda\rangle$

What does this mean?

A. We create an 'extremely unstable narrow width wave packet of width $\Gamma_{\vec{p}}$ resembling the state.

→ This is a 'resonance' in PP.

- How does this affect the particle?

$$E_p \to E_p + i\Gamma_p \longrightarrow \text{Will time evolve } e^{iE_p t} e^{-\Gamma_p t}$$

→ Lifetime $\sim (2\Gamma_p)^{-1}$

→ At $t > (2\Gamma_p)^{-1}$ we cannot expect a quasiparticle to be there anymore.

Important Question. With all these particles flying around AND decaying, how can we possibly know if we have a single particle?

- This is defined by the Quasiparticle weight : $\langle p_\lambda | \phi^\dagger(x) | \Omega \rangle = Z_p^{1/2} e^{ip.x} e^{-\Gamma_p t}$
- To describe a particle as a quasiparticle we NEED $E_{\vec{p}} > \Gamma_{\vec{p}}$

Energy Decay rate

- When $\Gamma_{\vec{p}}$ is small, we can describe particles in an interacting theory by multiplying (renormalizing) functions by the factor $Z_{\vec{p}}^{1/2}$

- Energies of interacting single particles vs free particles:

$$E_{\vec{p}} = (\vec{p}^2 + m^2)^{1/2}$$ Bare mass

$$E_{\vec{p}_\lambda} = (\vec{p}^2 + m_\lambda^2)^{1/2}$$

m_p^2 Interacting/Phy mass (Measured in exp)

Non-interacting:	$\|\boldsymbol{p}\rangle = \hat{a}^\dagger_{\boldsymbol{p}}\|0\rangle$	$\langle p\|\hat{\phi}(x)\|0\rangle = \mathrm{e}^{\mathrm{i}p\cdot x}$	m
Interacting:	$\|\boldsymbol{p}_\lambda\rangle = \hat{q}^\dagger_{\boldsymbol{p}}\|\Omega\rangle$	$\langle p_\lambda\|\hat{\phi}(x)\|\Omega\rangle = Z_{\boldsymbol{p}}^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}p\cdot x}\mathrm{e}^{-\Gamma_{\boldsymbol{p}}t}$	m_{P}

1.3 Propagator for a dressed particle

- In order to be able to do any computations with these quasi/renormalized particles, we need to have one of the most crucial quantities for it explicitly : The propagator!

$$G(x,y) = \langle\Omega| T\phi_H(x)\phi_H(y) |\Omega\rangle \tag{1}$$

- Chapter 7 of Peskin is DEDICATED to show the general form of this interacting propagator (without using any perturbation theory whatsoever). - I will upload my notes for that!

  We will just use the result here, and see that it fits well with the story we have built till now - which is - the propagator (built from $a_{\boldsymbol{p}}$ and $a^{\dagger}_{\boldsymbol{p}}$), will not only describe propagation of single particle states, but also decide the fate of multi particle states. It has the form

$$\tilde{G}(p) = \frac{\mathrm{i}Z_{\boldsymbol{p}}}{p^2 - m_{\mathrm{P}}^2 + \mathrm{i}\Gamma_{\boldsymbol{p}}} + \begin{pmatrix} \text{multiparticle} \\ \text{parts} \end{pmatrix} \tag{2}$$

where $Z_{\boldsymbol{p}}^{\frac{1}{2}} = \langle p_\lambda | p_\lambda\rangle \,\phi(0)\, |\Omega\rangle$ is the **amplitude for creating a single particle state** $|p_\lambda\rangle$ out of the interacting vacuum and m_P is the mass of the state $|\boldsymbol{p}_\lambda\rangle$

  - What is the role of $\Gamma_{\boldsymbol{p}}$? It is the inverse quasiparticle lifetime which took place of the usual infinitesimal factor ϵ. This makes even more sense as we are getting a propagator with a physical meaning completely and no need to worry about this ϵ factor.

  Remark : In the case of long-lived elementary particles, $Z_{\boldsymbol{p}} \to Z$, i.e. it is independent of the three momentum. The momentum dependence becomes more relevant for solids (metals).

Connection with experiment : Spectral density function

- One way to write full propagator makes contact with experiment uses a **spectral density function** $\rho(M^2)$.

  Example, for a scalar field theory we rewrite the propagator in terms of the free propagator

$$\Delta(x,y,M^2) = \frac{1}{p^2 - M^2 + i\epsilon} \tag{3}$$

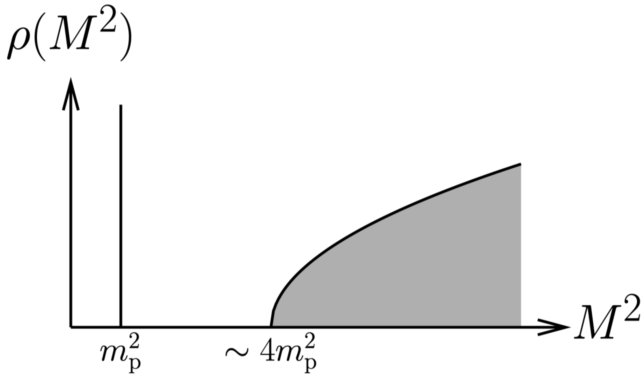
  in terms of this spectral density function as :

$$G(x,y) = \int_0^\infty \frac{\mathrm{d}M^2}{(2\pi)}\rho\left(M^2\right)\Delta\left(x,y,M^2\right) \tag{4}$$

- Now assume, we have the spectral density function,

$$\rho\left(M^2\right) = (2\pi)\delta\left(M^2 - m_{\mathrm{P}}^2\right)Z + \begin{pmatrix} \text{multiparticle} \\ \text{parts} \end{pmatrix}. \tag{5}$$

  which can be seen in



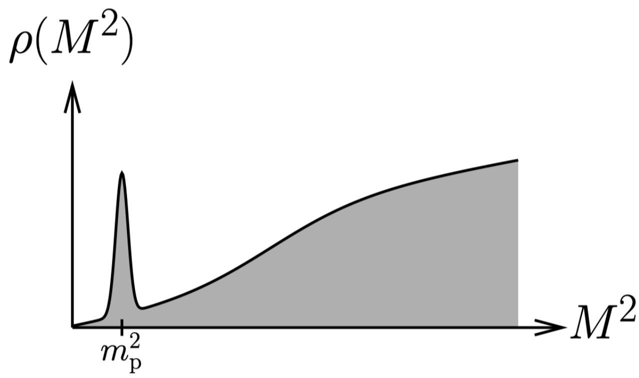
- It is made up of a single-particle peak at mass $M \equiv m_p$. This is the physical mass of a single particle in the interacting theory.
- It has a multiparticle continuum starting at $4m_p^2$, which is the minimal threshold for making two real particles in the COM frame.
- This kind of function is what you usually see in particle physics where creation of more than one particle is seen due to pair production.

Plug this into spectral density to get,

$$\tilde{G}(p) = \frac{\mathrm{i}Z}{p^2 - m_{\mathrm{P}}^2 + \mathrm{i}\epsilon} + \int_{\approx 4m_{\mathrm{P}}^2}^{\infty} \frac{\mathrm{d}M^2}{2\pi} \rho\left(M^2\right) \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\epsilon} \tag{6}$$

Here, we split the integral into two parts. Also note, the lifetime of the particle is ϵ^{-1}, which is effectively infinite.

► Now lets look at a different spectral density,



This has a quasiparticle peak with width $2\Gamma_{\boldsymbol{p}}$. This leads to a propagato,

$$\tilde{G}(p) = \frac{\mathrm{i}Z_{\boldsymbol{p}}}{p^2 - m_{\mathrm{P}}^2 + \mathrm{i}\Gamma_{\boldsymbol{p}}} + \left(\begin{array}{c} \text{multiparticle} \\ \text{parts} \end{array} \right), \tag{7}$$

where the quasiparticle now has a finite lifetime $(2\Gamma_{\boldsymbol{p}})^{-1}$ and we have also allowed the possibility of Z as a function of $\boldsymbol{p}$.

- Also talk about bounds states possibility between the one particle and multiparticle states.