Quantum Field Theory 2 – Exercise sheet 1

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Due date: 30 Apr/ 01 May 24

Thirty-one years ago Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes', he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function.' I said to him, 'You're crazy'. But he wasn't.
F. J. Dyson (1923–2020)

Problem 1 : Path integral in quantum mechanics [10 points]

Part (a) : Compute the measure for a canonical case

In the path integral formulation of quantum mechanics, we reach a point where we have the following expression,

$$\langle q'', t'' \mid q', t' \rangle = \int \mathcal{D}q \exp\left[i \int_{t'}^{t''} dt L(\dot{q}(t), q(t))\right] \tag{1}$$

Find an explicit formula for $\mathcal{D}q$ in the above expression. Your final formula will have the form,

$$\mathcal{D}q = C \prod_{j=1}^{N} dq_j \tag{2}$$

where C is the constant you need to compute.
Read the "**Recap for Part (a)**" section at the end of the problem before attempting the problem.

Part (b) : Path integral of a free particle

Assume that we have a free particle, i.e. $V(q) = 0$. For this case, evaluate the path integral in eq.(1) explicitly. Express your final answer in terms of q', t', q'', t'' and m. Restore $\hbar$ by dimensional analysis.
Hint : Integrate over q_1, then q_2, etc., and recognize a pattern.

Part (c)

Compute $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over momentum. Compare this result with your result in part (b). (Use the same Hamiltonian as in part (b))

Recap for Part (a) Here is a brief summary of where this term $\mathcal{D}q$ comes from :
Starting with

$$\langle q'', t'' \mid q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j, \bar{q}_j)\delta t} \tag{3}$$

where $\bar{q}_j = \frac{1}{2}(q_j + q_{j+1}), q_0 = q'$ and $q_{N+1} = q''$.

We can now define $\dot{q}_j \equiv \frac{q_{j+1}-q_j}{\delta t}$ and take the formal limit of $\delta t \to 0$ we get,

$$\langle q'', t'' \mid q', t' \rangle = \int \mathcal{D}q \mathcal{D}p \exp\left[i \int_{t'}^{t''} dt (p(t)\dot{q}(t) - H(p(t), q(t))) \right] \tag{4}$$

Assuming that the Hamiltonian is no more than quadratic in momenta then we can do the integral over dq as a Gaussian integral. We will assume that our Hamiltonian satisfies this form (true for many of the commonly used Hamiltonians in physics) is expressed as,

$$H(p, q) = \frac{1}{2m} p^2 + V(q) \tag{5}$$

If the term that is quadratic in p is independent of q (like in the case of our Hamiltonian), then the integration can be done over p and all of these prefactors are absorbed into

$$\mathcal{D}q : \text{ which we call as the } \textit{measure} \tag{6}$$

and the remaining integral will have the following form,

$$\langle q'', t'' \mid q', t' \rangle = \int \mathcal{D}q \exp\left[i \int_{t'}^{t''} dt L(\dot{q}(t), q(t)) \right] \tag{7}$$

Problem 2: Wick's theorem derivation using path integral formulation

[10 points]

Part (a) : Mathematical prelude

Let us define,

$$I_n = \int_{-\infty}^{\infty} dx\ x^{2n} e^{-\frac{1}{2}ax^2}. \tag{8}$$

(i) Explicitly compute I_0 (Ideally in less than 5 lines)

(ii) Using I_0, compute I_1

Hint : Do not explicitly compute I_1. Take some time and try to find a differential relation between I_0 and I_1. This is a foundational step in path integration!

Part (b) : Computing n-point correlation function

We define the n-point correlation function as,

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} dx\ x^n \mathrm{e}^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-\frac{1}{2}ax^2}}. \tag{9}$$

Calculate,

(i) $\langle x^2 \rangle$

(ii) $\langle x^4 \rangle$

(iii) $\langle x^n \rangle$

Hint : After computing the squared and quadratic correlation function, you generalize the result for an n-point correlation function. Essentially, to do it mathematically precisely you will need to use induction, but our goal here is just to get the result.

Part (c) : Diagramatically demystyfying n-point correlation function

We now want to diagrammatically make sense of the result of the expression $\langle x^n \rangle$ (the RHS you found in the previous part). To do this, represent each factor of $\frac{1}{a}$ as a line linking factors of x.

Part (d) : Generalzing our result to N dimensions

Now consider the N-dimensional vector $\mathbf{x}$ and the integral

$$\begin{aligned}\mathcal{K} &= \int \mathrm{d}x_1 \dots dx_N \mathrm{e}^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}+\mathbf{b}^{\mathrm{T}}\mathbf{x}} \\ &= \left(\frac{(2\pi)^N}{\det \mathbf{A}}\right)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{b}}.\end{aligned} \tag{10}$$

By differentiating with respect to components of the vector $\mathbf{b}$, and then setting $\mathbf{b} = 0$, show that

$$\langle x_i x_j \rangle = \frac{\int dx_1 \dots dx_N \; x_i x_j \mathrm{e}^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}}{\int dx_1 \dots dx_N \; \mathrm{e}^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}} = \left(\mathbf{A}^{-1}\right)_{ij}. \tag{11}$$

Part (e) : Voila! We have derived Wick's theorem

Using the result eq.(11), argue that

$$\langle x_i x_j x_k x_l \rangle = \left(\mathbf{A}^{-1}\right)_{ij} \left(\mathbf{A}^{-1}\right)_{kl} + \left(\mathbf{A}^{-1}\right)_{ik} \left(\mathbf{A}^{-1}\right)_{jl} + \left(\mathbf{A}^{-1}\right)_{il} \left(\mathbf{A}^{-1}\right)_{jk}. \tag{12}$$

and write down an expression for the general case

$$\langle x_i \dots x_z \rangle \tag{13}$$

Quantum Field Theory 2 – Exercise sheet 2

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Due date: 07/ 08 May 24

Sheldon : *Quantum physics makes me so happy.*
Leonard : *Yeah? I'm glad*
Sheldon : *It's like looking at the universe naked.*

- The Big Bang Theory (2007-2019)

Problem 1 : Path integral of complex scalar fields [6 points]

Problem statement

Show that for complex scalar fields

$$\int \mathcal{D}\phi^\star \mathcal{D}\phi \exp\left[i \int d^4x(\phi^\star M\phi + J^\star\phi + J\phi^\star)\right] = \mathcal{N}\frac{1}{\det M}\exp\left(iJM^{-1}J\right) \tag{1}$$

for some (infinite) constant $\mathcal{N}$

Problem 2 : Feynman rules for $\lambda\phi^4$ using path integrals [14 points]

Problem statement

The generating functional for a ϕ^4 theory is given by,

$$Z[J] = \left[\mathrm{e}^{-\frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4z \frac{\delta^4}{\delta J(z)^4}}\right] \mathcal{Z}_0[J] \tag{2}$$

where $\mathcal{Z}_0$ is the normalized generating functional for a *free scalar field.*
Our goal for this problem is to investigate the first-order term in the expansion of $Z[J]$, given by

$$Z_1[J] = \frac{\mathcal{N}_0}{\mathcal{N}}\left[-\frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4z \frac{\delta^4}{\delta J(z)^4}\right] \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)\Delta(x-y)J(y)} \tag{3}$$

The simple idea to reach our goal? Act on $\mathcal{Z}_0[J] = \mathrm{e}^{-\frac{1}{2}\int \mathrm{J}\Delta\mathrm{J}}$ with $\frac{\delta}{\delta J(z)}$ four times.

(a) : Attack #1 :

Verify that hitting $\mathcal{Z}_0[J]$ once with our functional differential operator gives us,

$$\frac{\delta \mathcal{Z}_0[J]}{\delta J(z)} = \left[-\int \mathrm{d}^4 y \Delta(z-y) J(y)\right] \mathcal{Z}_0[J] \tag{4}$$

(b) : Attack #2

Verify

$$\frac{\delta^2 \mathcal{Z}_0[J]}{\delta J(z)^2} = \left[-\Delta(z-z) + \left\{\int \mathrm{d}^4 y \Delta(z-y) J(y)\right\}^2\right] \mathcal{Z}_0[J] \tag{5}$$

(c) : Attack #3

Verify

$$\frac{\delta^3 \mathcal{Z}_0[J]}{\delta J(z)^3} = \left[3\Delta(z-z)\left\{\int \mathrm{d}^4 y \Delta(z-y) J(y)\right\}\right. \tag{6}$$

$$\left. - \left\{\int \mathrm{d}^4 y \Delta(z-y) J(y)\right\}^3\right] \mathcal{Z}_0[J] \tag{7}$$

(d) : Attack #4 (Final)

Verify

$$\begin{aligned}\frac{\delta^4 \mathcal{Z}_0[J]}{\delta J(z)^4} = & \left[3\Delta(z-z)^2 - 6\Delta(z-z)\left\{\int \mathrm{d}^4 y \Delta(z-y) J(y)\right\}^2\right. \\ & \left. + \left\{\int \mathrm{d}^4 y \Delta(z-y) J(y)\right\}^4\right] \mathcal{Z}_0[J].\end{aligned} \tag{8}$$

(e) : Tie a knot to all those pieces now

Expand the brackets, multiply by $-i\frac{\lambda}{4!}$ and integrate $\int \mathrm{d}^4 z$ to end up with the final result that $Z_1[J]$ is given by,

$$\begin{aligned} Z_1[J] = & -\frac{\mathcal{N}_0}{\mathcal{N}} \mathrm{i}\lambda \left[\frac{1}{8}\int \mathrm{d}^4 z \Delta(z-z)^2\right. \\ & - \frac{1}{4}\left\{\int \mathrm{d}^4 z \mathrm{d}^4 y_1 \mathrm{d}^4 y_2 \Delta(z-z)\Delta(z-y_1)J(y_1)\Delta(z-y_2)J(y_2)\right\} \\ & + \frac{1}{4!}\left\{\int \mathrm{d}^4 z \mathrm{d}^4 y_1 \mathrm{d}^4 y_2 \mathrm{d}^4 y_3 \mathrm{d}^4 y_4\right. \\ & \left.\left. \times \Delta(z-y_1)J(y_1)\Delta(z-y_2)J(y_2)\Delta(z-y_3)J(y_3)\Delta(z-y_4)J(y_4)\right\}\right] \mathcal{Z}_0[J]. \end{aligned} \tag{9}$$

by doing this we have terms in J of order $O(J) = 0, 2, 4$

(e) Feynman rules from path integral formulation

Interpret the previous result by drawing Feynman diagrams for each term.
Remark : This is the exact procedure that is used to work Feynman rules for a theory in the path integral formulation.

Quantum Field Theory 2 – Exercise sheet 3

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Institut für Theoretische Physik, Uni Heidelberg **Due date: 14/ 15 May 24**

The more success the quantum theory has, the sillier it looks.
- Albert Einstein (1879-1955)

Problem 1 : Ward identity for Compton scattering [10 points]

Problem statement

On sheet0 you have calculated the amplitude for Compton scattering $e^-(p)\,\gamma(k) \to e^-(p')\,\gamma(k')$ with the result

$$\mathcal{T} = e^2 \epsilon_\mu(k) \epsilon^*_\nu(k') A^{\mu\nu} \,, \tag{1}$$

where

$$A^{\mu\nu} = \overline{u}(p') \left[\frac{\gamma^\nu(\not{p} + \not{k} + m)\gamma^\mu}{(p+k)^2 - m^2} + \frac{\gamma^\mu(\not{p} - \not{k}' + m)\gamma^\nu}{(p-k')^2 - m^2} \right] u(p) \,. \tag{2}$$

Show that this amplitude satisfies the Ward Identity, i.e. $k_\mu A^{\mu\nu} = 0$ for on-shell electrons. You don't need to assume that the photons are on-shell, i.e. don't use $k^2 = k'^2 = 0$.

Problem 2 : Generating functional of connected correlation functions [10 points]

Problem Statement

We know now that taking functional derivatives of our generating functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\mathcal{L}[\phi] + J\phi \right) \right] \tag{3}$$

with respect to $J(x)$ produces correlation functions. But, the result that we get from this generating functional is not free of disconnected diagrams. In this problem we will formulate the *generating functional of connected correlation functions* which we define as

$$E[J] = \log Z[J] \tag{4}$$

(a) First derivative

Compute

$$\frac{\delta}{\delta J(x)} E\left[J\right] \tag{5}$$

and argue that it gives you,

$$\frac{\delta}{\delta J(x)} E\left[J\right] = i \left\langle\Omega\right| \phi(x) \left|\Omega\right\rangle_J \tag{6}$$

(b) Second derivative

Now we define,

$$\left\langle\Omega\right| \phi(x) \left|\Omega\right\rangle_J \equiv \left\langle\phi(x)\right\rangle \tag{7}$$

Compute

$$\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} \tag{8}$$

You will have two terms. Take your time and stare at these two terms and argue that what you have computed is

$$\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} = -\left\langle\phi(x)\phi(y)\right\rangle_{\text{conn}} \tag{9}$$

where the expectation on the right hand side is called as *connected correlator*.

(c) Third derivative

Sketch how the computation of

$$\frac{\delta^3 E[J]}{\delta J(x)\delta J(y)\delta J(z)} \tag{10}$$

would look like and argue that what you get is

$$\frac{\delta^3 E[J]}{\delta J(x)\delta J(y)\delta J(z)} = -i\left\langle\phi(x)\phi(y)\phi(z)\right\rangle_{\text{conn}} \tag{11}$$

(d) Generalized verison

Generalize what you have seen and write down the RHS of the following equation,

$$\frac{\delta^n E[J]}{\delta J(x_1)\cdots\delta J(x_n)} = \tag{12}$$

This answer convince you that the object $E\left[J\right]$ indeed is a generating function for *connected correlation functions*.

Quantum Field Theory 2 – Exercise sheet 5

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Due date: 28/ 29 May 24

> The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.
>
> \- Sidney Coleman (1937-2007)

Problem 5.1 : Renormalization of the one-loop effective action in ϕ^4 theory **[10 points]**

Problem statement

The one-loop effective action for ϕ^4 theory is given by

$$V_{\text{eff}}(\phi) = V_0(\phi) + \frac{1}{2}\int \frac{d^4p_E}{(2\pi)^4}\log\left[p_E^2 + m_{\text{eff}}^2\right], \tag{1}$$

where p_E denotes the Euclidean 4-momentum, m_{eff} and the tree-level action is given by,

$$V_0(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4!}\phi^4 + C_0 \tag{2}$$

Counterterms are defined by ,

$$m_0^2 = m^2(1+\delta_m), \quad \lambda_0 = \lambda(1+\delta_\lambda), \quad C_0 = C(1+\delta_C). \tag{3}$$

NOTE: Ideally we want all of you to use Mathematica to solve ALL parts of this problem

(a) Momentum integral with UV cutoff

Carry out the momentum integral with a UV cutoff $p_E^2 \leq \Lambda^2$, with $\Lambda^2 \gg m_{\text{eff}}^2$ to give you,

$$\frac{1}{2}\int \frac{d^4p_E}{(2\pi)^4}\log\left[p_E^2 + m_{\text{eff}}^2\right] = \frac{1}{16\pi^2}\left[-\frac{\Lambda^4}{8} + \frac{m_{\text{eff}}^2\Lambda^2}{4} + \frac{1}{4}\Lambda^4\log(\Lambda^2) + \frac{1}{4}m_{\text{eff}}^4\log\left(\frac{m_{\text{eff}}^2}{\Lambda^2}\right)\right] \tag{4}$$

Hint: Recall the identity

$$\int \frac{d^4k_E}{(2\pi)^4}f(k_E^2) = \frac{1}{8\pi^2}\int_0^\infty dk_E k_E^3 f(k_E^2) \tag{5}$$

(b) Renormalization conditions and counterterms

Impose the following renormalization conditions,

$$V_{\text{eff}}(\phi = 0) = C, \qquad V''_{\text{eff}}(\phi = 0) = m^2, \qquad V''''_{\text{eff}}(\phi = \mu) = \lambda, \tag{6}$$

and determine the three counterterms at one loop accuracy.

(c) Plug in counterterms

Plug the counterterms back into the effective action and show that it is of the form (which is finite in the limit of $\Lambda \to \infty$

$$V_{\text{eff}}(\phi) - V(\phi) = \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}\phi^2\right)^2 \log\left(1 + \frac{\lambda\phi^2}{2m^2}\right) - \frac{1}{64\pi^2}\left(\frac{\lambda}{2}m^2\phi^2 + \frac{3}{8}\lambda^2\phi^4\right) + c\frac{\lambda^2}{4!}\phi^4, \tag{7}$$

where $V(0)$ is define analogously to $V_0(\phi)$ with $m_0 \to m$, $\lambda_0 \to \lambda$, $C_0 \to C$. The lowercase c in the expression depends on λ and $\frac{\mu^2}{m^2}$.
At the end, determine c and verify that it satisfies $c \to 0$ for $\mu \to 0$ or equivalently $m \to \infty$)

(d) Final check

Show that $V_{\text{eff}}(\phi) \to V(\phi)$ for $m \to \infty$

Problem 5.2. Functional determinants in complex ϕ^4 theory [10 points]

Problem statement

Consider the Euclidean action,

$$S[\overline{\varphi}, \varphi] = \int d^4x_E \overline{\varphi}\left[-\partial_E^2 + m^2\right]\varphi + \frac{\lambda}{4}(\overline{\varphi}\varphi)^2 \tag{8}$$

which gives rise to the one-loop effective action,

$$\begin{aligned}\Gamma_{1-\text{loop}}(\overline{\phi}, \phi) &= -\ln \int \mathcal{D}\overline{\varphi}\mathcal{D}\varphi \exp\left(-\int d^4x_E \overline{\varphi}(x) S''[\overline{\phi}, \phi](x,x)\varphi(x)\right), \\ &= \ln \det S''[\overline{\phi}, \phi]\end{aligned} \tag{9}$$

which is formally given by the functional determinant of the operator,

$$S''[\overline{\phi}, \phi](x,y) \equiv \frac{\delta^2 S[\overline{\phi}, \phi]}{\delta\overline{\phi}(x)\delta\phi(y)}, \tag{10}$$

(a)

Express the one-loop effective action in terms of an infinite sum of Feynman diagrams, by directly calculating the path integral in perturbation theory around the free action (i.e., as it would be a theory with an interaction vertex $(\bar{\phi}\phi\bar{\varphi}\varphi)$)

(b)

Calculate the diagrams in terms of the free propagator $G(x,y) = (-\partial_E^2 + m^2)^{-1}$ and show that the effective action is given by (upto a ϕ-independent constant)

$$\Gamma_{1-\text{loop}}(\overline{\phi}, \phi) = -\sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} \text{tr} \left(G\overline{\phi}\phi \right)^n \tag{11}$$

where the trace is defined as

$$\text{tr} \left(G\overline{\phi}\phi \right)^n = \int d^4x_1 \dots d^4x_n \left[G(x_1, x_2)\overline{\phi}(x_2)\phi(x_2) \dots G(x_n, x_1)\overline{\phi}(x_1)\phi(x_1) \right]. \tag{12}$$

(c)

Show that this result agrees with the formal definition of the functional determinant upto a constant, i.e.

$$\Gamma_{1-\text{loop}}(\overline{\phi}, \phi) = \ln \det S''[\overline{\phi}, \phi] = \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \ln \left(1 + \lambda G\overline{\phi}\phi \right) + \text{constant}. \tag{13}$$

Quantum Field Theory 2 – Exercise sheet 6

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Institut für Theoretische Physik, Uni Heidelberg

Due date: 04/ 05 June 24

> Even before the advent of quantum mechanics, the properties of such numbers had already been examined by Hermann Grassmann. Although Grassmann's work was largely neglected during most of his life, the anticommuting numbers he invented are now known as Grassmann numbers. These numbers allow us to write down a coherent state for fermions and will allow us to write a path integral for fermions. The latter will be useful in our description of the quantum field of the electron.
>
> \- QFT for the gifted amatuer, Lancaster and Blundell (2014)

Problem 6.1 : Grassmannology [10 points]

Problem statement

Consider a quadratic form $\theta^T M\theta = \theta_i M_{ij}\theta_j$ where θ_i are Grassmann numbers and M is a an anti-symmetric $n \times n$ matrix of commuting numbers (possibly complex). Then consider the Gaussian integral

$$\int d\theta_1 \dots d\theta_n \, \exp\left(-\frac{1}{2}\theta^T M\theta\right), \tag{1}$$

where we take n even for simplicity.

(a)

Calculate the integral for $n = 2$.

(b)

Calculate the integral in a convenient basis, using the fact that every complex antisymmetric matrix can be brought to block-diagonal form via a unitary transformation V according to

$$V^T M V = \begin{pmatrix} 0 & +m_1 & \\ -m_1 & 0 & \\ & & \ddots \end{pmatrix} \tag{2}$$

where V is a unitary matrix, and each m_i is real and positive. Remember that for a general change of fermionic variables $\theta_i = U_{ij}\theta'_j$ one has

$$\int d\theta_1 \ldots d\theta_n X = \frac{1}{\det U} \int d\theta'_1 \ldots d\theta'_n X \,. \tag{3}$$

Problem 6.2 : Feynman rules of a Yukawa theory [10 points]

Problem Statement

Consider the theory of a real scalar interacting with a Dirac fermion given by the following Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi + y\phi\bar{\psi}\psi \tag{4}$$

(a)

Use functional methods (similar to the ones used in in problem 4.1 of our assignments) to derive the Feynman rules for the propagators and vertices involving the fermionic and scalar particles.

(b)

Additionally, draw the Feynman diagrams associated with one and two-loop corrections to propagators and vertices.

(c)

Use the previous results to determine the relative sign of the two tree-level amplitudes contributing to $\psi\psi \to \psi\psi$ scattering. What is the physical interpretation of this result?

Quantum Field Theory 2 – Exercise sheet 7

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Due date: 11/ 12 June 24

I have an equation, do you have one too?

- Paul Dirac (1902–1984), on meeting Richard Feynman

Problem 7.1 : Furry's theorem **[14 points]**

Problem statement

There is a theorem called Furry's theorem which states that,

$$\langle\Omega| T\{A_{\mu_1}(q_1)\dots A_{\mu_{n(q_n)}}\} |\Omega\rangle = 0 \tag{1}$$

for n odd. This is a consequence of invariance under the charge conjugation operation C.

(a)

Show that, in order for the action

$$S = \int d^4x\, \bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi, \tag{2}$$

to be invariant under charge conjugation

$$C: \quad \psi \to -i\gamma_2\psi^*, \quad C: \quad \psi^* \to -i\gamma_2\psi, \tag{3}$$

we need the photon field A^μ to be odd under charge conjugation

$$C: \quad A_\mu \to -A_\mu \tag{4}$$

Hint : The identity $\gamma^2\gamma^\mu\gamma^2 = (\gamma^\mu)^*$ will come in handy. Also, remember you are allowed to assume the integrate by parts to neglect boundary terms if needed.
Resource: This is a fantastic resource to keep open while trying to solve this `https://sites.ualberta.ca/~gingrich/courses/phys512/node64.html`

(b)

Prove Furry's theorem i.e. eq.(1) in QED using path integral formulation .

Hint : You have already proved the action is invariant under charge conjugation transformation. You are allowed to assume that the measure is invariant under charge conjugation (If it were not invariant, it would open a can of worms closely related to anomalies). You will require just these two facts to prove the theorem.

(c)

Does Furry's theorem hold if the photons are off-shell or just on-shell?

(d)

Draw the 1-loop diagrams for Compton scattering in QED. Which of the diagrams must vanish or cancel by Furry's theorem?

Problem 7.2 : Anticommutation [6 points]

Problem statement

Since Grassmann numbers anticommute, $\theta_1\theta_2\theta_1\theta_2 = 0$. But then,

i. Why does a term like $\bar{\psi}(x)\psi(x)\bar{\psi}(x)\psi(x)$ not automatically vanish?

ii. What about $(\bar{\psi}\psi)^5$, does it vanish? Explain your answer.

Quantum Field Theory 2 – Exercise sheet 8

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Due date: 18/ 19 June 24

The universe is an enormous direct product of representations of symmetry groups.
- Steven Weinberg (1933–2021).

Problem 8.1 : QED gauge invariance **[6 points]**

Problem statement

By adding a gauge fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to the QED Lagrangian, you keep the correlation functions of gauge invariant operators unchanged.

(a) Instead if I add a term $-\frac{1}{\xi}\left(\partial_\mu A^\mu\right)^4$, would the above statement still be true?

(b) What about if I add ξA_μ^2?

Problem 8.2 : Gauge invariance of scalar SU(2) Yang-Mills Theory **[14 points]**

(a) Covariant derivative transforms just like the field

Check that $D_\mu\vec{\phi}$,

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} - igA_\mu^a\tau^a\vec{\phi}, \tag{1}$$

transforms the same way as $\vec{\phi}$ transforms under infinitesimal SU(2) gauge transformation, given by

$$\vec{\phi}(x) \to \vec{\phi}(x) + i\alpha^a(x)\tau^a\vec{\phi}(x) \tag{2}$$

where τ^a are the SU(2) generators given by Pauli matrices, $\tau^a = \sigma^a/2$, provided the gauge field A_μ transforms as,

$$A_\mu^a(x) \to A_\mu^a(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - \epsilon^{abc}\alpha^b(x)A_\mu^c(x). \tag{3}$$

Hint : You might want to use the commutator relation $[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c$, where ϵ^{abc} is the totally antisymmetric Levi-Civita tensor with $\epsilon^{123} = 1$

(b) Gauge transformation of non-abelian field strength

Check that $F^a_{\mu\nu}$ transforms as

$$F^a_{\mu\nu} \to F^a_{\mu\nu} - \epsilon^{abc}\alpha^b F^c_{\mu\nu} \tag{4}$$

under infinitesimal SU(2) gauge transformations.
Hint: You might want to use Jacobi identity $\epsilon^{abd}\epsilon^{dce} + \epsilon^{bcd}\epsilon^{dae} + \epsilon^{cad}\epsilon^{dbe} = 0$.

(c) Gauge invariance of the SU(2) Yang-Mills Lagrangian

Use these results from the first two parts of this problem to show that,

$$\mathcal{L} = (D_\mu\vec{\phi})^\dagger D_\mu\vec{\phi} - \frac{1}{4}\sum_{a=1,2,3} F^a_{\mu\nu}(F^{\mu\nu})^a \tag{5}$$

is invariant under infinitesimal SU(2) gauge transformations.

Quantum Field Theory 2 – Exercise sheet 9

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Institut für Theoretische Physik, Uni Heidelberg

Due date: 25/ 26 June 24

Einstein's theory of General Relativity has a mathematical structure very similar to Yang-Mills theory.

- Chen-Ning Yang (1922 –)

Problem 9.1 : Noether current in Yang-Mills theory [10 points]

Problem statement

Consider the Yang-Mills Lagrangian given by,

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}\left(i\gamma^\mu D_\mu\right)\psi - m\bar{\psi}\psi \tag{1}$$

where $D_\mu = \partial_\mu - igA_\mu$ and $A_\mu \equiv A^a_\mu T^a$.

The equations of motion are,

$$\partial_\mu F^a_{\mu\nu} + gf^{abc}A^b_\mu F^c_{\mu\nu} = -g\bar{\psi}_i\gamma_\nu T^a_{ij}\psi_j \tag{2}$$

$$(i\partial\!\!\!/ - m)\psi_i = -gA\!\!\!/^a T^a_{ij}\psi_j. \tag{3}$$

We know $\mathcal{L}$ has a $SU(N)$ gauge symmetry (we studied this for SU(2) in the previous exercise sheet). Because of this, it also has a global symmetry under the following transformations (for infinitesimal α),

$$\psi_i \to \psi_i + i\alpha^a T^a_{ij}\psi_j \tag{4}$$

$$A^a_\mu \to A^a_\mu - f^{abc}\alpha^b A^c_\mu \tag{5}$$

(a) Deriving Noether current for Yang-Mills theory

Using Noether's theorem, show that the conserved current is given by,

$$J^a_\mu = -\bar{\psi}_i\gamma_\mu T^a_{ij}\psi_j + f^{abc}A^b_\nu F^c_{\mu\nu} \tag{6}$$

Hint:

$$J_\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}. \tag{7}$$

where ϕ_n can take both the matter fields $\psi_i, \overline{\psi}_i$ and the gauge fields A^a_μ. Here $\delta\phi = \phi' - \phi$ denotes the change of the field under an infinitesimal symmetry transformation.

(b) Showing Noether current is conserved in Yang-Mills theory

Show that the current is conserved on the equations of motion,

$$\partial_\mu J^a_\mu = 0 \tag{8}$$

Problem 9.2 : θ term in $SU(N)$ Yang-Mills theory

[10 points]

Problem statement

Consider the gauge-invariant dimension-4 "θ-term" in $SU(N)$ Yang-Mills theory

$$\mathcal{L}_\theta = \theta\, \varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta}\,, \tag{9}$$

with the non-abelian field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ where f^{abc} are the $SU(N)$ structure constants, and $\varepsilon^{\mu\nu\rho\sigma}$ denotes the totally anti-symmetric Levi-Civita tensor with $\varepsilon^{0123} = 1$.

Show that this term can be written as a total derivative, i.e.

$$\mathcal{L}_\theta = \theta\, \partial_\mu K^\mu \tag{10}$$

with the so-called Chern-Simons current

$$K^\mu = 2\, \varepsilon^{\mu\nu\alpha\beta} \left(A^a_\nu F^a_{\alpha\beta} - \frac{g}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta \right)\,. \tag{11}$$

This implies that the θ-term has no observable effects on the level of perturbation theory. However, it can give rise to non-perturbative effects, for example induce an electron dipole moment of the neutron within the Standard Model, which gives rise to the so-called Strong CP Problem.

Hint: In order to avoid index clutter you may want to switch to matrix notation, i.e. write $A_\mu = A^a_\mu T^a$ and $F_{\mu\nu} = F^a_{\mu\nu} T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ with $SU(N)$ generators T^a in the fundamental representation. Using $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ in the fundamental representation, the θ-term can be written in this notation as $\mathcal{L}_\theta = 2\theta\, \varepsilon^{\mu\nu\alpha\beta}\, \mathrm{tr}(F_{\mu\nu} F_{\alpha\beta})$. In a similar way one can write the Chern-Simons current as a trace of some suitable matrix expression.

Quantum Field Theory 2 – Exercise sheet 10

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Institut für Theoretische Physik, Uni Heidelberg

Due date: 02/ 03 July 24

> Nobody ever reads a paper in which someone has done an experiment involving photons with the footnote that says 'this experiment was done in Coulomb gauge'.
>
> - Sidney Coleman (1937 – 2007)

Problem 10.1 : Feynman rules for QCD vertices in Lorenz gauge [10p]

Problem statement

For the QCD Lagrangian in Lorenz gauge i.e. $G[A] = \partial_\mu A^a_\mu$,

$$\mathcal{L} = -\frac{1}{4}\big(F^a_{\mu\nu}\big)^2 - \frac{1}{2\xi}\big(\partial_\mu A^a_\mu\big)^2 + \big(\partial_\mu \bar{c}^a\big)\big(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu\big)c^c$$
$$+ \bar{\psi}_i\big(\delta_{ij} i\partial\!\!\!/ + g A\!\!\!/^a T^a_{ij} - m\delta_{ij}\big)\psi_j \tag{1}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

Derive the Feynman rules for the following vertices using functional derivatives:

(a) Quark-Quark-Gluon

(b) Gluon-Gluon-Gluon

(c) Ghost-Ghost-Gluon

Hint: Take all the momenta flowing into the vertex. This means that the derivative ∂_μ acting on a gluon with in-flowing momentum p and polarization ϵ_ν acts the part of the field proportional to $A_\nu \sim \epsilon_\nu e^{-ipx} a_p$, as in the vertex a gluon with momentum p is annihilated. Thus the derivative simply gives a factor $-ip_\mu$ in the vertex Feynman rule. For out-flowing momenta one would find the opposite sign.

Problem 10.2 : Feynman rules for QCD in Axial gauge [10p]

Problem statement

Axial gauges are defined by the gauge condition $G[A] = r_\mu A^a_\mu$ where r_μ is an arbitrary 4-tuple. For example, $r^\mu = (1, 0, 0, 0)$ gives the condition $A^a_0 = 0$ in the limit of $\xi \to 0$, which is the axial gauge in electromagnetism. When r^μ is light-like, i.e. $r^2 = 0$, this is called the light cone gauge.

(a) Calculate the resulting gauge-fixing and ghost terms in the Lagrangian for a general axial gauge.

(b) Show that the resulting gluon propagator is given by,

$$i\Pi^{\mu\nu ab}_{\text{axial}} = \frac{i}{p^2 + i\varepsilon}\left[-g^{\mu\nu} + \frac{r^\mu p^\nu + r^\nu p^\mu}{rp} - \frac{(r^2 + \xi p^2)p^\mu p^\nu}{(rp)^2}\right]\delta^{ab} \tag{3}$$

(c) Show that this propagator satisfies $p^\mu \Pi^{\mu\nu} = 0$ for onshell p^μ and $r^\mu \Pi^{\mu\nu} = 0$ for $\xi = 0$

(d) Calculate the ghost-ghost-gluon vertex and show that ghosts decouple for $\xi = 0$

Quantum Field Theory 2 – Exercise sheet 11

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Institut für Theoretische Physik, Uni Heidelberg

Due date: 09/ 10 July 24

> The mathematics clearly called for a set of underlying elementary objects. At that time we needed three types of them elementary objects that could be combined three at a time in different ways to make all the heavy particles we knew. I needed a name for them and called them quarks, after the taunting cry of the gulls, "Three quarks for Muster mark," from Finnegan's Wake by the Irish writer James Joyce.
>
> \- Murray Gell-Mann (1929 – 2019) on choosing the term 'quark'

Problem 11.1 : Nilpotency of BRST transformations [10p]

Problem statement

Prove that the BRST variation of the BRST variation of the gauge field vanishes, i.e.

$$\delta_B(\delta_B A^a_\mu) = 0. \tag{1}$$

Recall the following relations,

$$\delta_B A^a_\mu(x) \equiv D^{ab}_\mu c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_\mu(x) c^b(x) \tag{2}$$

$$\delta_B c^c(x) = -\frac{1}{2} g f^{abc} c^a(x) c^b(x) \tag{3}$$

Hint: Remember that δ_B is fermionic.

Problem 11.2 : Gauge Transformations and BRST Cohomology [10p]

Problem statement

The creation operator for transversal photon can be written as,

$$a^\dagger_i(\vec{k}) = \frac{i}{\sqrt{2\omega_{\vec{k}}}} \epsilon^\mu_i(\vec{k}) \int d^3x e^{-ikx} \overleftrightarrow{\partial_0} A_\mu(x) \tag{4}$$

where index $i \in \{1, 2\}$ and $\omega_{\vec{k}} = |\vec{k}|$.
Consider the state $a^\dagger_i |\psi\rangle$, where $|\psi\rangle$ is in the BRST cohomology. Define a gauge-transformed polarization vector,

$$\tilde{\epsilon}^\mu_i(\vec{k}) = \epsilon^\mu_i(\vec{k}) + c k^\mu \tag{5}$$

where c is a constant, and a corresponding creation operator $\tilde{a}_i^\dagger(\vec{k})$. Show that,

$$\tilde{a}_i^\dagger \left|\psi\right\rangle = a_i^\dagger(k) \left|\psi\right\rangle + Q_{\mathrm{B}} \left|\chi\right\rangle \tag{6}$$

which implies that $\tilde{a}_i^\dagger(\vec{k}) \left|\psi\right\rangle$ and $a_i^\dagger(\vec{k}) \left|\psi\right\rangle$ represent the same element of the cohomology, and hence are physically equivalent. Find the state $\left|\chi\right\rangle$.

Quantum Field Theory 2 – Exercise sheet 12

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Due date: 16/ 17 July 24

> It turns out that the obvious generalization of this idiotically simple manipulation gets rid of all of the infinities for any field theory with polynomial interactions, to any order in perturbation theory.
>
> - Sidney Coleman (1937–2007) on renormalization

Problem 12 : Gauge boson self-energy in non-Abelian gauge theory [20p]

Problem statement

Consider a non-Abelian gauge theory described by a Yang-Mills Lagrangian with n_f fermions.
The aim of the present problem is to calculate the g^2 contribution to the gauge-boson self-energy and to show explicitly that, in agreement with gauge invariance, it is transverse with respect to the gauge-boson incoming four-momentum. In order to evaluate the Feynman amplitudes, employ dimensional regularization in $d = 4 - \epsilon$ dimensions.

Note : You are expected to do all the computations in Feynman gauge $\xi \to 1$. Ambitious people may attempt it for an arbitrary ξ.

(a)

Draw the relevant Feynman diagrams contributing to the gauge-boson self-energy to order g^2. Assume that the external gauge-boson legs are characterized by the gauge- and Lorentz- indices a, μ and b, ν respectively, and by the four-momentum p^α.

(b)

Use the known results from QED to evaluate the contribution to the fermion loop. Appropriately isolate the divergent term of the corresponding amplitude. Is this contribution transverse with respect to the four momentum p^μ, i.e., is it proportional to $p^2 g^{\mu\nu} - p^\mu p^\nu$?

(c)

Compute two diagrams contributing to the self-energy that involves only gauge bosons.
Remark : Although the so-called seagull diagram involving the four-gluon vertex can be shown to formally vanish in four-dimensions, it is convenient to artificially include its

contribution. Its evaluation involves the integral $\int d^d k k^{-2}$, which can be easily taken by multiplying and dividing the integrate by $(p+k)^2$ and by using the same technique employed to evaluate the other amplitude involving virtual gauge-bosons.

(d)

Verify explicitly that the sum of the contributions considered so far has two problems :

(a) It features a quadratic divergence, i.e., a divergence for $d = 2$ in the language of dimensional regularization

(b) It is not transverse with respect to the external gauge-boson four-momentum

(e)

Compute the contribution due to the ghost fields and show explicitly that it fixes both problems in (d).

Hint : Use the identity,

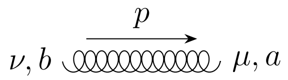
$$\int_0^1 dx\, x f(x(1-x)) = \int_0^1 dx (1-x) f(x(1-x)), \tag{1}$$

which implies that,

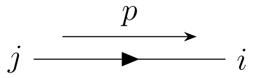
$$\int_0^1 dx\, x f(x(1-x)) = \int_0^1 dx \frac{1}{2}(x+1-x) f(x(1-x)) = \frac{1}{2}\int_0^1 dx f(x(1-x)) \tag{2}$$

for an arbitrary function f to simplify the Lorentz tensor structure of this contribution.

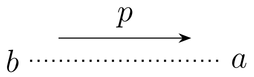
Feynman rules for QCD



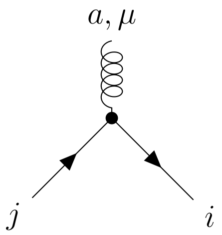
$$= \frac{-i\delta^{ab}}{p^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right) \tag{3}$$



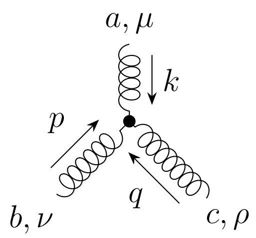
$$= \frac{i}{\not{p} - m + i\epsilon}\delta^{ij} \tag{4}$$



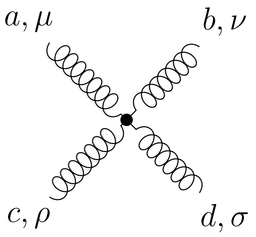
$$= \frac{i\delta^{ab}}{p^2} \tag{5}$$



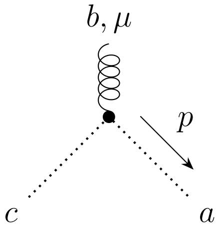
$$= ig\gamma^\mu T^a_{ij} \tag{6}$$



$$= gf^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right] \tag{7}$$



$$\begin{aligned} = -ig^2[&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned} \tag{8}$$



$$= -gf^{abc}p^\mu \tag{9}$$

Loop integrals and Dimensional Regularization

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}. \tag{10}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \tag{11}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n - \frac{d}{2} - 1} \tag{12}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n - \frac{d}{2} - 1} \tag{13}$$

$$\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \Big(\frac{1}{\Delta}\Big)^{2 - \frac{d}{2}} = \frac{1}{(4\pi)^2} \Big(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\Big) \tag{14}$$

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d \tag{15}$$

Quantum Field Theory 2 – MOCK EXAM (v2)

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Institut für Theoretische Physik, Uni Heidelberg

Due date: 23/24 July 2024

Problem 1 : BRST symmetry **[10p]**

Consider the action of a non-Abelian gauge theory including the gauge-fixing term and ghost terms via the Faddeev-Popov method,

$$S = \int_x \left\{ -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \bar{\psi}(i\gamma_\mu D_\mu - m)\psi + \partial_\mu \bar{c}^a D^{ab}_\mu c^b \right\}, \tag{1}$$

with $D_\mu = \partial_\mu - igA_\mu$ and $D^{ab}_\mu c^b = \partial_\mu c^a + gf^{abc}A^b_\mu c^c$.

(a) Show that the above action is equivalent to

$$S = \int_x \left\{ -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\gamma_\mu D_\mu - m)\psi + \frac{\xi}{2}B^a B^a + B^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ac}_\mu c^c \right\} \tag{2}$$

upon employing the equation of motion for the Nakanishi-Lautrup field B^a.

(b) Demonstrate that the Dirac action,

$$S_D\left[\psi, \bar{\psi}, A\right] = \int_x \bar{\psi}\left(i\gamma_\mu D_\mu - m\right)\psi \tag{3}$$

is invariant under a BRST transformation

$$\delta_{\mathrm{B}} A^a_\mu = D^{ac}_\mu c^c, \quad \delta_{\mathrm{B}} c^a = -\frac{1}{2}gf^{abc}c^b c^c \tag{4}$$

$$\delta_{\mathrm{B}} \bar{c}^a = B^a, \quad \delta_{\mathrm{B}} B^a = 0, \quad \delta_{\mathrm{B}} \psi_i = igc^a T^a_{ij}\psi_j \tag{5}$$

Problem 2 : Non-linear gauge fixing **[10p]**

Consider the action for pure electrodynamics

$$S_{\mathrm{ED}} = \int_x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \tag{6}$$

Write down the gauge fixed and ghost Lagrangian using the Faddeev-Popov method for the non-linear gauge,

$$G\left[A\right] = \partial_\mu A_\mu + \frac{1}{2}\eta(A_\mu)^2 \tag{7}$$

(a) Invert the kinetic operators that appear in the action to find the propagators for the photon field and for the ghost field.

(b) Is the ghost field decoupled in this gauge?

Hint :

$$S_{\text{gauge-fixing}} = -\int_x \frac{1}{2\xi} G[A]G[A] \tag{8}$$

$$S_{\text{ghost}} = \int_{x,y} \bar{c}(x)M(x,y)c(y) \tag{9}$$

$$\text{with } M(x,y) = \partial_{\mu,y}\frac{\delta G[A(x)]}{\delta A_\mu(y)} \tag{10}$$

Problem 3 : Grassmann integrals [5p]

Calculate the following integrals for Grassmann variables $\theta_i, i = 1,2,3$ in terms of the elements of the 3×3 matrix M:

$$\int (d\theta_1 d\theta_2 d\theta_3) \exp^{-\frac{1}{2}\sum_{i,j}\theta_i M_{ij}\theta_j} \tag{11}$$

$$\int (d\theta_1 d\theta_2 d\theta_3)\, \theta_1 \exp^{-\frac{1}{2}\sum_{i,j}\theta_i M_{ij}\theta_j} \tag{12}$$

where the sums run over all indices, i.e. $i, j = 1, 2, 3$.

Problem 4 : Simplifying SU(N) Generator Commutators [5p]

Simplify the expression

$$\sum_a [T^a T^a, T^b] \tag{13}$$

for generators T^a in the adjoint representation of an SU(N) group, i.e. $(T^a)^{bc} = -if^{abc}$ with f^{abc} being the structure constants.

Problem 5 : Gauge-Scalar Interactions in SU(N) Yang-Mills Theory [5p]

Construct the Lagrangian for scalar-gauge boson couplings for a complex scalar field in the fundamental representation by starting from the free scalar kinetic Lagrangian and requiring invariance under local SU(N) transformations.

Problem 6 : Short answers [5p]

(a) Explain why one has to add gauge-fixing terms in order to quantize YM theories

(b) What is the generating functional of 1PI diagrams?

(c) What is the significance of the Dyson-Schwinger equations and their relation to the classical equations of motion?