* I really need to recall the source of these notes. I had either followed some book or lectures!

FADDEV–POPOV QUANTISATION.

- Deal w/ redundancy in Gauge theories. (Recall for $U(1)$ what's that)

Abelian — Non-Abelian

↳ Both have same procedure to be quantized, but Abelian MUCH EASIER.
(We will soonish see why!)

Start Quantization for $U(1)$ gauge boson.

- [Euclidean space]: Go from $(M) \to (\mathbb{R}^4 \equiv E)$ by a Wick rotation (Essentially a COV)

$$x^0 = x_0 = t = -ix_4 = -ix^4 \quad , \quad \partial_0 = \frac{\partial}{\partial x^0} = i\frac{\partial}{\partial x^4} = i\partial_4$$

↳ In Quantizing $U(1)$ gauge boson, we will deal w/ A_μ, which transforms like x_μ. So we also rotate A_0 like $x_0 = -t$: $A_0 \equiv iA_4$

↳ $$A_\mu = (A_0, A_j) = (iA_4, A_j) \quad , \quad A_\mu A^\mu = -A_0^2 + A_j^2 = -(iA_4)^2 + A_j^2 = A_4^2 + A_j^2$$

- [E space in E_i]:

$$E_i^{(M)} = F_{0i} = \partial_0 A_i - \partial_i A_0 = i\partial_4 A_i - \partial_i (iA_4) = i\underbrace{(\partial_4 A_i - \partial_i A_4)} = i\, F_{4i} = i\, E_i^{(E)}$$

Similarly, $F^{0i} = -iF^{4i}$

- [E space massless $U(1)$ action]

{Why not transform this? Wick rotation does $M \mapsto E$ transformation by just COV in zeroth component}

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{0i}F^{0i} - \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{4}F_{4i}F^{4i} - \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{4}F^{(E)}_{\mu\nu}F^{(E)\mu\nu}$$

↳ This is just the $\mathcal{L}$, we need to see the action

$$i\underbrace{\int d^4x}_{-i\,\partial^4 x_E}\,\mathcal{L}^{(E)}_{EM} = \frac{1}{4}F^{(E)}_{\mu\nu}F^{(E)\mu\nu} = \frac{1}{2}\left[(E_i^{(E)})^2 + (B_i^{(E)})^2\right]$$

(Only $dx^0 \to -i\,dx^4$)

$$\left(= \frac{1}{2}\left[(B_i)^2 - (E_i)^2\right]\right)$$

* We will drop the (E) index now, but will be working in E (Till I say we change)

• **[The Action and the path integral]** : (Integrating by parts)
& GAUGE FIXING idea

$$S_{EM}[A] = \int d^4x \left[-\frac{1}{2}A_\mu\left(\delta_{\mu\nu}\partial^2 - \partial_\nu\partial_\mu\right)A_\nu\right]$$ → This is the Euclidean action.

$$Z = \int DA_\mu(x)\; e^{-S_{EM}[A]} \quad -\ (1)$$

(i) $A'_\mu \neq A''_\mu - \frac{1}{e}\partial_\mu \chi$

(ii) $A'_\mu = A'_\mu - \frac{1}{e}\partial_\mu \chi$

Remind everyone where this comes from.

Recall the idea of Path $\int$
:- You are integrating over all the possible configuration $A_\mu(x)$
- The sum of all these gives you the probability amplitude.
- Classically, the one minimizing the action is the only one contributing.

↳ In eq(1), there are two types of contributions to $A_\mu(x)$ [The function over which we are taking the path $\int$]

(i) This is the desired one as we are $\int$ over all <u>physically inequivalent</u> configs of A_μ that give the quantum behaviour of the field.

(ii) This is (i) + physically equivalent configs of A_μ

↳ Not good! As we have a copy of A_μ for EACH choice of χ which end up in the $\int$.

↳ Try 'fixing' this :

$$\int DA_\mu(x) = \underbrace{\int d\chi}_{\int \text{over Gauges}} \underbrace{\int DA_\mu^{(GF)}(x)}_{\int \text{over physically relevant fields (GF)}} \quad -\ (2)$$

→ Gauge-fixed.

- (Doing this is a challenging task - even for Abelian, but much more for Non-abelian)
- (The whole business of GF can be naively boiled down to : <u>Avoid double-counting</u>)

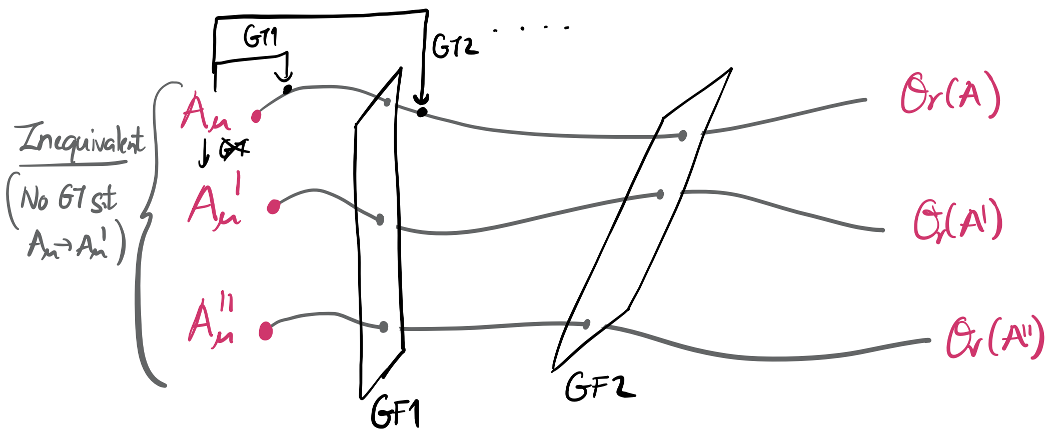
↳ In eq.(2), if we accomplish doing this, we remove all dependence of χ from the path $\int$

↳ The $\int d\chi$ just becomes a multiplicative factor in Z. (The interpretation is that : This ↗ value is the volume of the internal space defined by $U(1)$)

- We know choosing $\partial_\mu A_\mu = 0$ (Lorenz gauge) removes the redundancy. Our goal is to see why does it do that!

↳ Start w/ a more general gauge : $\boxed{\partial_\mu A_\mu = C(x)}$ — 3

For a given specific field configuration A_μ, the orbit $Or(A)$ is the set of all other configurations that can be obtained through A_μ (through GT)



↳ In the 'space of all possible' GF conditions, the 'good' are the ones that only have one point of intersection between this 'space' & $Or(A)$ (for each ineq config)

↳ For Abelian theories, we can assume this intersection is unique (for perturbative regime)
For Non Abelian, there is a problem of Gribov copies (Very hard to get 1 intersection per orbit)
(We will cross this bridge when we come to it, right now lets focus on quantizing $U(1)$)

↳ GF is finding these single intersection planes, & $\int d\chi$ is the volume you get by $\int$ of these planes.

- (PARAMETRIZE the orbit)

Assume that we are travelling through the orbit using the following transformation

$$\boxed{A_\mu(x) \to {}^{\chi}A_\mu(x) \equiv A_\mu(x) + \partial_\mu \chi(x)}$$

→ ${}^{\chi}A_\mu(x)$ is any point in the orbit that I can get starting

Taking ∂^μ, $\quad \underbrace{\partial^\mu {}^\chi A_\mu(x)}_{\text{If} \to C(x)} = \partial^\mu A_\mu(x) + \partial^2 \chi(x)$ | from $A^\mu(x)$

$$\therefore \quad \boxed{\partial^2 \chi(x) = -\partial^\mu A_\mu(x) + C(x)}$$

Basically, this is the value of χ that brings us into GF condition, $\partial^\mu A_\mu = C$

This is the transformation we need such that eq.3. is satisfied.

- **[PROVE IDENTITY]**

↳ Use the two variables ${}^\chi A_\mu(x)$, $\chi(x)$ to prove the following identity:

$$\boxed{\int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta\left(-\partial_\mu({}^\chi A_\mu(y)) + C(y)\right) = \frac{1}{\mathrm{Det}(-\partial^2)}}$$

Why? Because if this is true, we will have found the identity

$$1 = \mathrm{Det}(\partial^2) \overbrace{\int \prod_{x \in \mathbb{R}^4} d\chi(x)}^{\int d\chi \text{ Integrating over all the gauges.}} \overbrace{\prod_{y \in \mathbb{R}^4} \delta\left(-\partial_\mu({}^\chi A_\mu(y)) + C(y)\right)}^{\delta[-\partial_\mu A + C]\ :\ \delta \text{ functionl s.t. the derivative of } A \text{ must be } C\ \forall y \in \mathbb{R}^4}$$

(F

↳ If we place this identity inside our path integral of A, through δ we can impose eq.3 $\left(\partial_\mu A_\mu = C(x)\right)$

- Let us demonstrate the above point:

$$-\partial_\mu({}^\chi A_\mu) + C = -\partial^2 \chi - \partial_\mu A^\mu + C = -\partial^2 \chi + \partial^2 \chi^{(A)}$$

It is a const. f^n.

This is the function that puts us in the point in the Or that crosses the GF condition for a value of C — IMP

Make a change of variables: $\chi \to \chi - \chi^{(A)}$ (Inside the integral)

$$\int \prod_{x \in \mathbb{R}^4} d\chi(x) \prod_{y \in \mathbb{R}^4} \delta\Big[\underbrace{-\partial_\mu({}^\chi A_\mu(y)) + C(y)}_{G({}^\chi A_\mu)}\Big] \overset{?}{=} \int \underbrace{\prod_{x \in \mathbb{R}^4} d\chi(x)}_{\text{No change in the measure.}} \prod_{y \in \mathbb{R}^4} \delta\left[-\partial^2 \chi(y)\right]$$

$$G({}^\chi A_\mu(y)) = -\partial_\mu({}^\chi A_\mu(y)) + C(y) = -\partial^2 \chi(y)$$

$$\frac{\delta G({}^{x}A_\mu(y))}{\delta x(x)} = -\partial^2\,\delta(x-y) \equiv -\partial^2(x,y)$$

↳ This Op is acting on $x(y)$ inside the δ function.

↳ This $\widehat{Op}$ is like a Jacobian for a change of variables:

$x \to G({}^{x}A_\mu)$ (Lets go to the discrete version to see this)

$$\int \prod_{i=1}^{n} dx_i \prod_{j=1}^{N} \delta(\overset{\nearrow\,\text{Some op}}{\Delta_{ij}}x_j) = \int d^n x\; \delta^n(\underbrace{\Delta\vec{x}}_{\equiv\,\vec{n}})$$

$\equiv \vec{n},\quad n_i = \Delta_{ij}x_j$

$d^n n = \mathrm{Det}(\Delta)\, d^n x$

$$= \int \frac{d^n n}{\mathrm{Det}(\Delta)}\, \delta^n(\vec{n})$$

$$= \frac{1}{\mathrm{Det}(\Delta)} \underbrace{\int d^n n\; \delta^n(\vec{n})}_{=1}$$, just $\int$ over δ f^n.

$$= \frac{1}{\mathrm{Det}\,\Delta}$$

↳ The functional version:

$$\int \prod_{x\in\mathbb{R}^4} dx \prod_{y\in\mathbb{R}^4} \delta(\underbrace{-\partial^2 x(y)}) = \int \prod_x \frac{dn(x)}{\mathrm{Det}\left[\frac{\delta G}{\delta x}\right]} \prod_y \delta(n(y))$$

$n(y) = -\partial^2 x(y)$

$dn = \mathrm{Det}(-\partial^2)\, dx$

$\hookrightarrow = \mathrm{Det}\left[\frac{\delta G}{\delta x}\right]$

$$= \frac{1}{\mathrm{Det}[\,..\,]} \int \overbrace{\prod_x dn(x)}^{d^n n}\; \overbrace{\prod_y \delta(n(y))}^{\delta^n(\,..\,)}$$

$$= \frac{1}{\mathrm{Det}[\,..\,]}$$

$\therefore$
$$\boxed{\int \prod_x dx \prod_y \delta(-\partial^2 x(y)) = \frac{1}{\mathrm{Det}[-\partial^2]} = \frac{1}{\mathrm{Det}\left[\frac{\delta G}{\delta x}\right]}}$$

- If I plug this identity inside my path $\int$, I would have GF^d it.
- What is the effect of this Dirac δ that I am putting in, in terms of a Lagrangian term.
  - ↳ Can I convert this to a term in the Lagrangian to see physically the effect?
  - ↳ Need one more identity:

The $\int$ is a Gaussian.

$$\underbrace{N(\alpha)}_{\text{Normalization, i.e. } \frac{1}{\int\ldots}} \int Dc \; e^{-\frac{1}{2\alpha}\int d^n x \, c^2(x)} = 1$$

Just a #

- We have two identities $=1$. Lets multiply them:

$$N(\alpha)\int Dc \; e^{-\frac{1}{2\alpha}\int d^n x \, c^2(x)} \; Det(-\partial^2)\int D\chi \; \delta[-\partial_\mu {}^\chi A_\mu + c] = 1$$

- Now we use the δ to do the path $\int$ in c.

$$\int D\chi \; N(\alpha) \; e^{-\frac{1}{2\alpha}\int d^n x \, (\partial_\mu {}^\chi A_\mu(x))^2} \, Det(-\partial^2) = 1$$

- Insert this identity for the path $\int$ in field A.

$$\int DA_\mu \; e^{-S[A]} \; \underbrace{\mathcal{O}[A]}_{\text{Some observable / operator} \to \text{Anything that I can calculate in this theory.}}$$

$$= \int DA_\mu \; e^{-S[A]} \; \mathcal{O}[A] \int D\chi \; N(\alpha) \; e^{-\frac{1}{2\alpha}\int d^n x \, (\partial_\mu {}^\chi A_\mu(x))^2} \, Det(-\partial^2)$$

- Few interesting things that I can do.

(1st, VV IMP) : Det does not depend on χ or A. (See this from ⊕) $\nearrow \frac{\delta G}{\delta \chi}$ does not depend on A.

↳ This is the point where if we had non-abelian theory, would be different.

In non-abelian theories $\frac{\delta G}{\delta \chi}$ will depend on A. (Need extra steps to deal w/ this determinant)

$$\int DA_\mu \, e^{-S[A]} \mathcal{O}[A] = \mathrm{Det}(-\partial^2) N(\alpha) \int D\chi \int DA_\mu \, e^{-S[A] - \frac{1}{2\alpha}\int d^4x \, \underbrace{(\partial_\mu {}^\chi A_\mu(x))^2}_{\circledast}} \mathcal{O}[A]$$

- Basically, I am calculating the observable w/ this new action.

- Bit nicer : ${}^\chi A = A + \partial_\mu \chi$ or $\underline{A} = {}^\chi A - \partial_\mu \chi$

  Origin of orbit.

- Major point : Action is Gauge invariant. So we can remove or add χ at will.

  $S[A] = S[{}^\chi A]$ & $\mathcal{O}[A] = \mathcal{O}[{}^\chi A]$

- Big assumption : $\int^n$ measure is also Gauge invariant. $DA = D^\chi A$.

  (Neither guaranteed, nor obvious)

  ↳ True & can be proved for most Gauge symmetries.

  ↳ Few where these fail, you get Anomaly.

  ↳ QED not anamalous so it is true & can be proved.

⇒ Bring in χ indices everywhere where invariant. Then remove from everywhere, especially $\circledast$ where its not invariant.

$$\int DA_\mu \, e^{-S[A]} \mathcal{O}[A] = \underbrace{\mathrm{Det}(-\partial^2) N(\alpha)}_{\text{Other \#s.}} \underbrace{\int D\chi}_{\text{Just a \# / Big Volume.}} \int DA_\mu \, e^{-S[A] - \frac{1}{2\alpha}\int d^4x \, (\partial_\mu A_\mu(x))^2} \mathcal{O}[A]$$

- Expectation value will be written.

$$\langle \mathcal{O}[A] \rangle = \frac{\int DA \;\; \mathcal{O}[A] \;\; e^{-S[A]}}{\int DA \;\; e^{-S[A]}} \;\; \frac{\uparrow}{\uparrow} = \frac{\# \int DA \;\; \mathcal{O}[A] \;\; e^{-S_{Eff}[A]}}{\# \;\; \int DA \;\; e^{-S_{Eff}[A]}}$$

- Now $\mathcal{L}_{Eff} = \frac{1}{2} F_{\mu\nu}^2 + \underbrace{\frac{1}{2\alpha} (\partial_\mu A_\mu)^2}_{\text{Gauge fixing term. } \mathcal{L}_{GF}.}$

↳ You could just add in this term by hand saying you are not really changing the Classical theory, but maybe you are changing the quantum theory. → We showed that we aren't.

↳ Assumption was $DA = DA^{\chi}$. There will be anomalous theories → Symmetry of classical but not Quantum.

Photon propagator

$$S_{EFF} = \int d^4x \left[-\frac{1}{2} A_\mu (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) A_\nu \right] - \underbrace{\frac{1}{2\alpha} A_\mu \partial_\mu \partial_\nu A_\nu}_{\text{GF part}}$$

$$\boxed{= \frac{1}{2}\int d^4x \; A_\mu(x) \underbrace{\left(-\delta_{\mu\nu}\partial^2 + \left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)}_{\text{Invertible.}} A_\nu(x)}$$

$$\left(G^{(0)}\right)^{-1}_{\mu\nu}(x)$$

$$= \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k)\, e^{ikx} \left(-\delta_{\mu\nu}\partial^2 + \left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right) \int \frac{d^4k'}{(2\pi)^4} \tilde{A}_\nu(k')\, e^{ik'x}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} (2\pi)^4 \delta^4(k+k') \; \tilde{A}_\mu(k) \left(\delta_{\mu\nu} k'^2 - \left(1 - \frac{1}{\alpha} k'_\mu k'_\nu\right)\right) \tilde{A}_\nu(k')$$

Then $k' \to k$

$$= \boxed{\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(-k) \underbrace{\left(\delta_{\mu\nu}k^2 - \left(1-\frac{1}{\alpha}\right)k_\mu k_\nu\right)}_{\left(G^{(0)^{-1}}(k)\right)_{\mu\nu}} A_\nu(k)}$$

$$\left(\delta_{\mu\nu}k^2 - \left(1-\frac{1}{\alpha}\right)k_\mu k_\nu\right) G^{(0)}_{\nu\rho}(k) = \delta_{\mu\rho}$$

$$\boxed{G^{(0)}_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)}$$

Photon prop in euclidean mom space.

Depends on α. Related to GF in a complicated way

$\alpha = 1$ Feynman gauge.

· We did $\partial_\mu A_\mu = c$ & then $\int Dc \; e^{-\frac{1}{2\alpha}\int d^4x \, c^2(x)}$