Causal structure in General Relativity

Rohan Kulkarni

Insititute of Theoretical Physics
University of Leipzig

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1. Basic Definitions.
   - Basics
   - Definitions of Future and Pasts
   - Definitions of Causal relations between events.

2. Causality
   - Different orders of Causality on a Spacetime.
   - Global dependence and Cauchy Surfaces
   - Cauchy Horizon

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Types of vectors on a Manifold

We have a metric tensor g_{ab} associated with *4-dimensional manifold* (Spacetime) as a solution to the Einstein's equation.The discussion throughout the talk will be on arbitrary spacetimes which we will denote by (M, g).By *arbitrary* we mean that we won't attempt to impose Einstein's equation on g_{ab}.

- This metric g defines a scalar product at any point $p \in M$ between two vectors in the tangent space T_pM.
- Along with the scalar product it also defines a norm of any single vector in T_pM. Based on the sign of the norm we have classified vectors at a point in three types:
    1. $g_{ab}v^av^b < 0$: Timelike
    2. $g_{ab}v^av^b = 0$: Lightlike or Null
    3. $g_{ab}v^av^b > 0$: Spacelike

Remark Time orientable

At every event event $p \in M$ the tangent space T_pM is isomorphic to the Minkowski spacetime. We have light cones through every event.

- A light cone passing through the origin of T_pM is defined as the Light cone of *p.*
  - Light cone of p is a subset of T_pM and not of M.
- These light cones have a designation of *future* and *past.*
- If a *continuous designation* of future and past can be made over our whole manifold M then we call our spacetime (M, g) **time orientable.**
- All the properties/definitions/theorems that we will state will be assumed to be on a Time orientable spacetime.

Character of Curves

Definition

Character of a point at a curve:

A curve $\gamma(t)$ at a point $p_0 = \gamma(t_0)$ is **timelike, null or spacelike** if its *tangent* vector $\dot{\gamma}(t_0)$ is *timelike, null or spacelike* respectively.

Future and past directed curve:

A *timelike* or a *null* curve which lies in the ***future/past half*** of the light cone is called **future/past directed curve** respectively .

Global character of curve:

A curve γ which is *timelike/spacelike/null* at every event is a **timelike/spacelike/null curve** respectively.

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Chronological future and past.

Definition

Chronological future/past of an event and a set of events:
For a given event $p \in M$, the **Chronological future/past** is defined by the set of all points that can be reached by a *future/past directed* ***timelike*** *curve* which starts at p. The set of these points is denoted by $\mathrm{I}^{\pm}(p)$ respectively.

$$\mathrm{I}^{\pm}(p) = \{q \in M \,|\, \exists \text{ Future/Past directed timelike curve } \lambda(t) \text{ with } \lambda(0) = p \text{ and } \lambda(1) = q\}$$

Causal future and past.

Definition

Causal future/past of an event and set of events:

For a given event $p \in M$, the **Causal future/past** is defined by the set of all points that can be reached by a *future/past directed* ***causal*** *curve* which starts at p. The set of these points is denoted by $\mathrm{J}^{\pm}(p)$ respectively.

$$\mathrm{J}^{\pm}(p) = \{q \in M \mid \exists \text{Future/Past directed timelike or null curve } \lambda(t) \text{ with } \lambda(0) = p \text{ and } \lambda(1) = q\}$$

Chronological/Causal Future/Past of set of points.

Remark

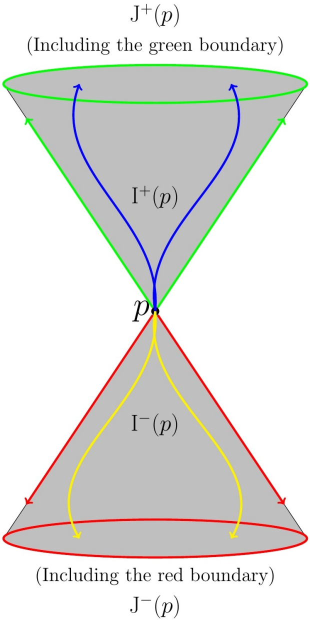
For a set of points U, the chronological future/past is defined as follows :

$$\mathrm{I}^{\pm}(U) = \bigcup_{p \in U} \mathrm{I}^{\pm}(p)$$

For a set of points U, the causal future/past is defined as follows

$$\mathrm{J}^{\pm}(U) = \bigcup_{p \in U} \mathrm{J}^{\pm}(\mathrm{p})$$

Definitions of Causal relations between events.



Intextendible future causal curve.

Definition

Future endpoint of a curve.

Let λ be a future directed causal curve in M. We say that $p \in M$ is a ***future endpoint*** of λ if for every neighbourhood O or p $\exists t_0$ such that $\lambda(t) \in O$ for all $t > t_0$.

Thus by *hausdorff property of M* we can have at most one future endpoint.

Future inextendible curve

A causal curve γ_c is called ***future inextendible*** if it does not have a **future endpoint.**

Intextendible past causal curve.

Definition

Past endpoint of a curve

Let λ be a past directed causal curve in M. We say that $p \in M$ is a ***past endpoint*** of λ if for every neighbourhood O or p $\exists t_0$ such that $\lambda(t) \in O$ for all $t < t_0$.

Thus by *hausdorff property of M* we can have at most one past endpoint.

Past inextendible curve

A causal curve γ_c is called ***past inextendible*** it does not have a **past endpoint.**

Achronal set.

Definition

Achronal sets

Achronal sets $\mathcal{A}$ are subsets of spacetime M that hold the property $\mathcal{A} \cap I^{+}(\mathcal{A}) = \varnothing$.

Remark.

$\rightarrow$ Intuitively : In these sets there are no such events which lie in the future of another event in the set. Imagine a set of events S (points in a spacetime) and any future event of these points does not belong to the set S.

$\rightarrow$ No two events in an *Achronal set* are *Causally connected* to each other i.e. "No two events in an *Achronal set* can be connected to each other by *null or timelike* curves."

Examples of Achronal set.

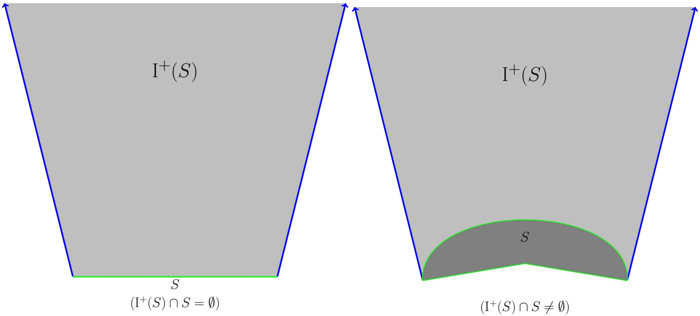


Figure: Achronal and Not Achronal set in 2-D Minkowski spacetime.

Edge of an Achronal set

Definition

Edge of an Achronal set

Let $\mathcal{A}$ be an *Achronal set*. Let $\partial\mathcal{A}$ be the *Edge* of the *Achronal set*. Then $\partial\mathcal{A}$ is the subset of all the events $p \in \mathcal{A}$ such that every neighbourhood $\mathcal{U} \in M$ of event p contains at least a point $p_+ \in \mathrm{I}^+(p)$ and $p_- \in \mathrm{I}^-(p)$ and a *timelike curve* γ_T connecting p_+ and p_- where $\gamma_T \cap \mathcal{A} = \emptyset$.

Causal spacetime.

Definition

Causal spacetime

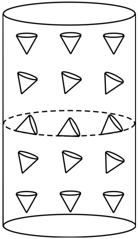
A Spacetime (M, g) is called **causal** if it *does not contain* a ***closed causal curve.*** I like to call it “Simply Causal” as we are going to have different orders of causal.

Drawbacks

- We still allow our spacetime to be arbitrarily closed to being causal.
- This could allow *timelike curves* which are not “closed” but arbitrarily close to being “*closed*”.
- .We don’t like this because this allows us intuitively to “nearly” go back in time.

Cylindrical spacetime

Figure: Cylindrical Spacetime



- If Cones become even a little bit more horizontal then intuitively we will be having timelike curves which are tending to become closed timelike curves.
- Such a spacetime is causal but barely causal i.e. “not strongly causal".

Strong Causality

Definition

Strongly Causal

(M,g) is called **Strongly causal** if $\forall p$, $p \in M$ and $\forall$ neighborhoods $\mathcal{U}$ of p there is a neighborhood $V \subseteq \mathcal{U}$ such that no *causal* curve γ intersects V more than once.

- If (M, g) is not *Strongly causal* $\Rightarrow$There exists a causal curve γ which comes arbitrarily close to intersecting itself.
- We require **strong causality** to keep causal curves at least a finite distance from intersecting themselves.
- Spacetimes in which the events from the future can influence their past i.e. spacetimes in which there are closed causal curves **do not** satisfy **Strong causality.** From this point of view strong causality seems like a sensible physical requirement.

Need stronger Causality

- Now it seems that the definition of *Strong causality* should be enough to do physics on our spacetime as it gives us an sensible condition that events from the future cannot effect events in their past. But there is still a small issue.
- We have not done anything to avoid this condition of non-existence of closed timelike curves in case of small perturbations. Lets deal with that now and we will have a causal structure of a spacetime with which we can work.
- For that we will define ***Stably Causal.*** But before denying that lets consider our problem with ***Strong causality*** and then define ***stable causality*** as a result of that problem.

Mathematically how and why to define Stable causality.

Problem

Arbitrarily small perturbations in the metric could allow causal curves to self intersect. Find a condition to avoid this.(We will define this condition as the necessary condition for Stable causality.)

Solution:

- Consider a perturbing metric g through :

$$g_{mn} \rightarrow \tilde{g}_{mn} = g_{mn} - \omega_m \omega_n$$

  with a *timelike cotangent vector field* ω_m.
  - So we have two metrics on the same differentiable manifold, g_{mn} and $\tilde{g}_{mn}$. g_{mn} is Lorrentzian / Pseudo Riemannian.
- Notice : $\tilde{g}_{mn}$ still has the same signature but "Light cones are now wider for $\tilde{g}_{mn}$."

- Notice : $\tilde{g}_{mn}$ still has the same signature but "Light cones are now wider for $\tilde{g}_{mn}$."
- How to see this?
- Compare $g_{mn}v^m v^n$ and $\tilde{g}_{mn}v^m v^n$. (Calculate the length of square of tangent vector v from the tangent vector field v^a at point p with respect to the two metrics)

$$\tilde{g}_{mn}v^m v^n = g_{mn}v^m v^n - v^m \omega_m v^n \omega_m \qquad (1)$$

- Now look at the second term on the right hand side carefully if $v^m w_n = \alpha \in \mathbb{R}$

$$v^m \omega_n v^m \omega_n = \alpha \cdot \alpha = \alpha^2 > 0$$

- Giving us

$$\tilde{g}_{mn}v^m v^n < g_{mn}v^m v^n$$

How does this solve our problem?

- Thus, it is easier for vectors v^a to have $||v|| < 0$ i.e. to be timelike or null w.r.t $\tilde{g}_{mn}$ than w.r.t g_{mn}.
- "Some vector that is *spacelike* with respect to g_{mn} ***maybe*** *timelike with respect to* $\tilde{g}_{mn}$".
- $(M, \tilde{g})$ spacetime has all *causal curves* of (M, g) plus more curves.
  - i.e. $\{\text{Causal curves of } (M, g)\} \subseteq \{\text{Causal curves of } (M, \tilde{g})\}$

So we have solved our problem by perturbing our metric and still maintaining the condition for causality. This condition where the perturbed metric also stays sensibly causal is know as **Stably Causal.**

Stably Causal Spacetime

Definition

(M, g) is called *Stably causal* if there exists a timelike covector field ω_a such that $(M, \tilde{g})$ is also causal. $(\tilde{g}_{mn} = g_{mn} - \omega_m \omega_n)$.

Theorem

If(M, g) is ***stably causal*** *then it* ***implies*** *that (M, g) is* ***strongly causal.***
Proof can be found in Wald's Book.

Theorem

(M, g) is stably causal iff there exists a differentiable function $f \in C^{\infty}(M, \mathbb{R})$ such that $\nabla^a f$ is a ***past directed timelike vector field*** *or $-\nabla^a f$ is a* ***future directed timelike vector field****.*

Remark

Intuitively, this means that f *or more precisely the level sets of* f can be viewed as a **cosmic clock**. (Not a unique one as we can have more than one f satisfying our conditions.)

Recall

We defined something called "*Time orientability of a spacetime*" at the start of this section.
(M, g) is *time orientable* iff there exists a *past/future pointing smooth timelike vector field.* What separates this from the theorem above the remark is that here the *smooth timelike vector field* need not be a *Gradient field.*

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Domain of dependence of Sets

Definition

Future domain dependence of set S. $(D^+(S))$

$S \subseteq M$ is a *closed achronal set*. **Future domain of dependence of S** is defined as

$$D^+(S) := \{p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S\}$$

Definition

Past domain dependence of set S. $(D^{-}(S))$

$$D^{-}(S) := \{p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S\}$$

Remark

The set of events p that affect only S.

Domain of dependence

Definition

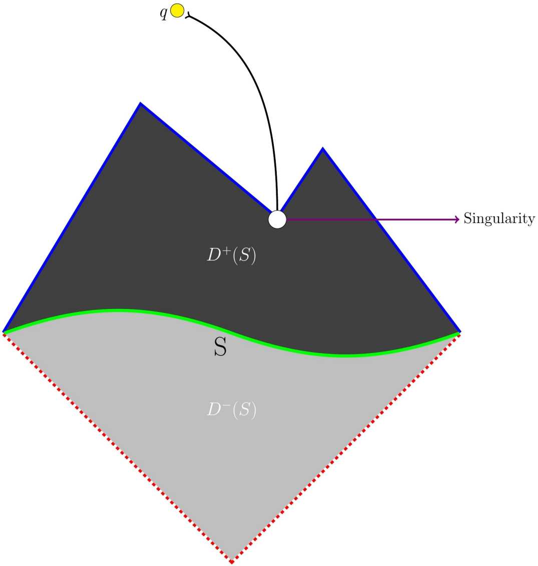
Domain of Dependence. ($D(S)$)
This is just defined as the union of the *future and past domain of dependence.*

$$D(S) = D^{+}(S) \cup D^{-}(S)$$

Remark

The total domain of dependence of set S *i.e.* both past and future events affected by S.

Example of domain of dependence



Remark about the figure

- One question we want to ask is, “Why $q \notin D^+(M)$?”
  - For q , some *past inextendible curve* does not intersect S because it gets stuck at the *singularity.* In other words we can say, “q is affected by events in the *shadow* of the singularity”.

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Future/Past Cauchy Horizon

Definition

Future Cauchy horizon of S. $(H^{+}(S))$

$$H^{+}(S) := \overline{D^{+}(S)} - I^{-}\left(D^{+}(S)\right)$$

- $\overline{D^{+}(S)}$: Closed set of future domain of dependence of S.
- $I^{-}(D^{+}(S))$: Chronological past of the future domain of dependence of S.

Definition

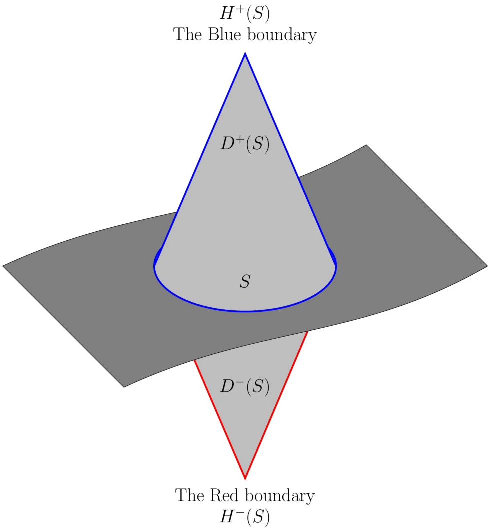
Past Cauchy Horizon : $H^{-}(S)$

$$H^{-}(S) := \overline{D^{-}(S)} - \overline{I^{+}(D^{-}(S))}$$

Remark

Set of earliest events that affect only S.

Diagram of Cauchy Horizons and Domains of dependence



Example

Remark

$H^{\pm}(S)$ is ***achronal.*** This is quite obvious as no two events in $H^{\pm}(S)$ are *causally connected* to each other.

Cauchy Horizon

Definition

Cauchy Surface.

A *closed, achronal set S* is called a "**Cauchy Surface**", if its *full cauchy horizon vanishes.* That is, if

1. $H(S) = \emptyset$
2. $\dot{D}(S) = \emptyset$
3. $D(S) = M$

- Look at point **3** carefully, the domain of dependence of S is the entire manifold M (Spacetime).
- What this physically means is that, if you know what happened on S (Initial conditions with suitable evolution laws) then you can predict what can happened on the entire manifold M (i.e. the entire spacetime).

- This is the reason why ***Cauchy surfaces*** are so important. If the conditions on a *Cauchy surface* are known then everything on M can be predicted.
- Since a *Cauchy surface* is *Achronal* , it can be viewed as an instant in time. (As no two events are connected causally on an *Achronal set*)

Definition

Globally Hyperbolic

A spacetime which has a **Cauchy surface** is called ***Globally hyperbolic spacetime.***

Summary of most important points.

- We first setup some conditions on our Spacetime so we can do sensible physics on it. By sensible I mean, we got rid of closed timelike curves and avoided events from future to affect their past.
- Then we defind domains and dependences and Cauchy surfaces. Having a Cauchy surface in our Spacetime makes it globally hyperbolic which enables us to know everything that has happened or will happen in our spacetime by investigating just a slice of our spacetime.
- We need a lot of these definitions along with some definitions of Asymptotic flatness in order to define a ***Black hole*** which I believe will be done in one of the upcoming talks.

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