Harmonic oscillator: $V(x) = \frac{1}{2} m\omega^2 x^2$ **ROHAN KULKARNI**

$= \frac{1}{2}x^2$

Let $m = \omega = 1 = \hbar$

$\hat{p}|\psi\rangle = \frac{1}{i}\frac{d|\psi\rangle}{dx}$

$\hat{x}|\psi\rangle = x|\psi\rangle$

We know the TISE: $\hat{H}|\psi\rangle = E|\psi\rangle$

$$\left(\frac{\hat{p}^2}{2} + V\right)|\psi\rangle = E|\psi\rangle$$

$$\left(\frac{1}{i}\frac{d}{dx}\right)^2|\psi\rangle + V|\psi\rangle = E|\psi\rangle$$

$$\rightarrow \quad -\frac{d^2|\psi\rangle}{dx^2} + V|\psi\rangle = E|\psi\rangle$$

[TISE for H.O.]

So our hamiltonian in the smallest form is: $H = \frac{1}{2}[\hat{p}^2 + \hat{x}^2]$

Arbitrary step 1: Define $\hat{a}^\dagger$ & $\hat{a}$. (PART I) → Buildup.

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \quad \& \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

Arbitrary step 2: Calculate $\hat{a}\hat{a}^\dagger$.

$$\hat{a}\hat{a}^\dagger = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})(\hat{x} - i\hat{p}) = \frac{1}{2}\left(\hat{x}^2 - i\hat{x}\hat{p} + i\hat{p}\hat{x} - \hat{p}^2\right)$$

$$= \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 - i(\hat{x}\hat{p} - \hat{p}\hat{x})\right) = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 - i[\hat{x},\hat{p}]\right)$$

What is $[\hat{x},\hat{p}]$? (Use a test function as in $L^2(\mathbb{R}) \to \infty$ dim $\mathcal{H}$)

$$\rightarrow [\hat{x},\hat{p}]|\psi_{test}\rangle = \hat{x}\hat{p}|\psi_{test}\rangle - \hat{p}\hat{x}|\psi_{test}\rangle = \hat{x}\frac{1}{i}\frac{d|\psi_t\rangle}{dx} - \hat{p}x|\psi_t\rangle$$

$$= \frac{1}{i}x\frac{d|\psi_t\rangle}{dx} - \frac{1}{i}\frac{d}{dx}\left(x|\psi_t\rangle\right) = \cancel{\frac{x}{i}\frac{d|\psi_t\rangle}{dx}} - \cancel{\frac{x}{i}\frac{d|\psi_t\rangle}{dx}} - \frac{1}{i}|\psi_t\rangle$$

$$= -\frac{1}{i}|\psi_t\rangle = i|\psi_t\rangle \qquad \therefore [\hat{x},\hat{p}] = i\hbar \rightarrow \hbar = 1 \text{ in this calc}^n.$$

$$\hat{a}\hat{a}^{+} = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 + \hat{I}\right)$$

Careful observation:

$$\hat{H} = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2\right) \quad \& \quad \hat{a}\hat{a}^{+} = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 + \hat{I}\right)$$

$$\therefore \quad \boxed{\hat{H} = \left(\hat{a}\hat{a}^{+} - \frac{\hat{I}}{2}\right)}$$

Bookkeeping calculation: (If needed) $[\hat{a}, \hat{a}^{+}] = \hat{I}$, $\Rightarrow [\hat{a}^{+}, \hat{a}] = -\hat{I}$

$$\boxed{\hat{H} = \left(\hat{a}^{+}\hat{a} + \frac{\hat{I}}{2}\right)}$$

Bookkeeping calculation (2): TISE in terms of $\hat{a}$ & $\hat{a}^{+}$.

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\left(\hat{a}\hat{a}^{+} - \frac{\hat{I}}{2}\right)|\psi\rangle = \left(\hat{a}^{+}\hat{a} + \frac{\hat{I}}{2}\right)|\psi\rangle = E|\psi\rangle$$

Technique, manipulate the LHS of the blue equation such that $\hat{H}$ action $|\psi\rangle$ first as we know $\hat{H}|\psi\rangle = E|\psi\rangle$

Proving 2 IMP Properties of $\hat{a}, \hat{a}^{+}$:

i) Claim $|\psi\rangle$ satisfies $\hat{H}|\psi\rangle = E|\psi\rangle$, & then prove $(\hat{a}^{+}|\psi\rangle)$ satisfies the S.E with energy $(E+1)$ $\{(E+\hbar\omega)\}$ $\hbar\omega \to =1$

i.e. $\hat{H}(\hat{a}^{+}|\psi\rangle) = (E+1)\,\hat{a}^{+}|\psi\rangle$

→ Proof:

Take this $\hat{a}^{+}$ out common. as $\hat{I}\hat{a}^{+} = \hat{a}^{+}\hat{I}$

$$\hat{H}(\hat{a}^{+}|\psi\rangle) = \left(\hat{a}^{+}\hat{a} + \frac{\hat{I}}{2}\right)(\hat{a}^{+}|\psi\rangle) = \left(\hat{a}^{+}\hat{a}\hat{a}^{+} + \frac{\hat{I}}{2}\hat{a}^{+}\right)|\psi\rangle$$

$$= \hat{a}^{+}\left(\hat{a}\hat{a}^{+} + \frac{\hat{I}}{2}\right)|\psi\rangle = \hat{a}^{+}\left(\hat{a}^{+}\hat{a} + \hat{I} + \frac{\hat{I}}{2}\right)|\psi\rangle$$

We know $[\hat{a}, \hat{a}^{+}] = \hat{I} = \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}$, ∴ This step → $\hat{a}\hat{a}^{+} = \hat{a}^{+}\hat{a} + \hat{I}$

$$= \hat{a}^{+}\left(\hat{H}+\hat{I}\right)|\psi\rangle = \hat{a}^{+}\left(\hat{H}|\psi\rangle+\hat{I}|\psi\rangle\right)$$

$$= \hat{a}^{+}(E+1)|\psi\rangle = (E+1)\,\underline{\hat{a}^{+}|\psi\rangle}$$

$$\therefore \hat{H}\left(\hat{a}^{+}|\psi\rangle\right) = (E+1)\left(\hat{a}^{+}|\psi\rangle\right)$$

ii) Claim $|\psi\rangle$ satisfies $\hat{H}|\psi\rangle = E|\psi\rangle$ and then prove $\hat{a}|\psi\rangle$ satisfies the S.E. with energy $(E-1)$. i.e.

$$\hat{H}\left(\hat{a}|\psi\rangle\right) = (E-1)\left(\hat{a}|\psi\rangle\right)$$

LHS: $\hat{H}\left(\hat{a}|\psi\rangle\right) = \left(\hat{a}\hat{a}^{+}-\frac{\hat{I}}{2}\right)\left(\hat{a}|\psi\rangle\right) = \left(\hat{a}\hat{a}^{+}\hat{a}-\frac{\hat{I}}{2}\hat{a}\right)|\psi\rangle$

$$= \left(\hat{a}\hat{a}^{+}\hat{a}-\frac{\hat{a}\hat{I}}{2}\right)|\psi\rangle = \hat{a}\left(\hat{a}^{+}\hat{a}-\frac{\hat{I}}{2}\right)|\psi\rangle = \hat{a}\left(\hat{a}\hat{a}^{+}-\hat{I}-\frac{\hat{I}}{2}\right)|\psi\rangle$$

$$= \hat{a}\left(\hat{H}-\hat{I}\right)|\psi\rangle = \hat{a}\left(\hat{H}|\psi\rangle-\hat{I}|\psi\rangle\right) = \hat{a}(E-1)|\psi\rangle$$

$$= (E-1)\left(\hat{a}|\psi\rangle\right)$$

$$\therefore \hat{H}\left(\hat{a}|\psi\rangle\right) = (E-1)\left(\hat{a}|\psi\rangle\right)$$

→ ∴ We now see where they get their names as 'raising' and 'lowering' operator.

PART II: DERIVING NORMALIZED Eigenstates $|\psi_n\rangle$

→ Start with the boundary condition:

Let $|\psi_0\rangle$ be the lowest state, then define $\underline{\hat{a}|\psi_0\rangle = 0}$

→ We can find $|\psi_0\rangle$ (The ground state by plugging in the definition of $\hat{a}$)

LHS: $\hat{a}|\psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{x}+i\hat{p}\right)|\psi\rangle = \frac{1}{\sqrt{2}}\left(x|\psi_0\rangle+\frac{i}{i}\frac{d|\psi\rangle}{dx}\right) = \text{RHS} = 0.$

$\therefore$ the O.D.E is $\quad x|\psi_0\rangle + \frac{d}{dx}|\psi_0\rangle = 0 \Rightarrow x|\psi_0\rangle = -\frac{d|\psi_0\rangle}{dx}$

$$\Rightarrow -x\,dx = \frac{d|\psi_0\rangle}{|\psi_0\rangle} \xRightarrow{\int} \frac{-x^2}{2} + C = \ln|\psi_0\rangle$$

Exponentiating

$$\Rightarrow |\psi_0\rangle = e^{-\frac{x^2}{2}+C} = e^{-\frac{x^2}{2}} e^C = A e^{-\frac{x^2}{2}}.$$

$\rightarrow$ Normalize this : $\| \psi_0 \|^2 = 1$

$$\Rightarrow \langle \psi_0 | \psi_0 \rangle = 1 \Rightarrow \int A e^{-\frac{x^2}{2}} A e^{-\frac{x^2}{2}} dx$$

$$= A^2 \int e^{-x^2} dx = A^2 \sqrt{\pi}$$

$$\therefore A^2 = \frac{1}{\sqrt{\pi}} \Rightarrow A = \frac{1}{\sqrt[4]{\pi}} \qquad \therefore |\psi_0\rangle = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{x^2}{2}}$$

Now lets find the ground state energy : (Using the fact $\hat{a}|\psi_0\rangle = 0$)

$$\hat{H}|\psi_0\rangle = \left(\hat{a}^+\hat{a} + \frac{\hat{I}}{2}\right)|\psi_0\rangle = \left(\hat{a}^+\hat{a}|\psi_0\rangle + \frac{\hat{I}}{2}|\psi_0\rangle\right) = \frac{\hat{I}}{2}|\psi_0\rangle = \frac{1}{2}|\psi_0\rangle$$

$\therefore$ If we think of it as $\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle \Rightarrow \underline{E_0 = \frac{1}{2}}.$

Now we have our foot placed on the ground state!

Using # from part 1 we can deduce that:

We can apply raising operators repeatedly to generate excited states, increasing the energy by 1 with every step.

#' $\boxed{|\psi_n\rangle = A_n (\hat{a}^+)^n |\psi_0\rangle \quad ; \quad E_n = \left(n + \frac{1}{2}\right)}$

Normalization constant.

Starts with 0 for Harmonic oscillator.

$\rightarrow$ Next step: Calculate A_n algebraically!

PTO.

We know, $\hat{a}^\dagger|\psi_n\rangle$ and $\hat{a}|\psi_n\rangle$ is proportional to $|\psi_{n+1}\rangle$ and $|\psi_{n-1}\rangle$ respectively.

$$\therefore \quad \hat{a}^\dagger|\psi_n\rangle = c_n|\psi_{n+1}\rangle \quad \& \quad \hat{a}|\psi_n\rangle = d_n|\psi_{n-1}\rangle \quad \#''$$

Show that $\hat{a}^\dagger$ and $\hat{a}$ are hermitian conjugates: i.e. $\psi, \phi \underset{x\to\pm\infty}{\longrightarrow} 0$ — $(*)$

Prove that $\langle \psi|\hat{a}^\dagger\phi\rangle = \langle \hat{a}\psi|\phi\rangle$ where $\psi \,\&\, \phi \in L^2(\mathbb{R})$

Lets prove from LHS to RHS and claim that RHS → LHS is done similarly.

Proof: $\langle\psi|\hat{a}^\dagger\phi\rangle \underset{\text{Def}^n \text{ of inner product in } L^2(\mathbb{R})}{=} \int_{-\infty}^{\infty}\psi^* \hat{a}^\dagger \phi\, dx = \int_{-\infty}^{\infty}\psi^* \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})\phi\, dx$

$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\psi^*\left(\hat{x} - \frac{i}{i}\frac{d}{dx}\right)\phi\, dx = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\psi^*\left(x\phi - \frac{d\phi}{dx}\right)dx$$

$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\left(\psi^* x\phi - \psi^*\frac{d\phi}{dx}\right)dx = \frac{1}{\sqrt{2}}\left[\int_{-\infty}^{\infty}\psi^* x\phi\, dx - \int_{-\infty}^{\infty}\psi^*\frac{d\phi}{dx}dx\right] \rightarrow (*)'$$

$$\rightarrow \int_{-\infty}^{\infty}\psi^*\frac{d\phi}{dx}dx = \underbrace{\psi^*\phi\Big|_{-\infty}^{\infty}}_{=0 \text{ from } (*)} - \int_{-\infty}^{\infty}\left(\frac{d\psi}{dx}\right)^*\phi\, dx = -\int_{-\infty}^{\infty}\left(\frac{d\psi}{dx}\right)^*\phi\, dx$$

Plug in.

[Identity used for integration by parts: $\int_a^b f\frac{dg}{dx}dx = fg\Big|_a^b - \int_a^b \frac{df}{dx}g\, dx$]

After plugging in

$$(*)' = \frac{1}{\sqrt{2}}\left[\int_{-\infty}^{\infty}\psi^* x\phi\, dx + \int_{-\infty}^{\infty}\left(\frac{d\psi}{dx}\right)^*\phi\, dx\right] = \frac{1}{\sqrt{2}}\left[\int_{-\infty}^{\infty}x\psi^*\phi\, dx + \dots\right]$$

$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\left[x\psi^*\phi + \left(\frac{d\psi}{dx}\right)^*\phi\right]dx = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}(\hat{x} + i\hat{p})\psi^*\phi\, dx$$

$$= \int_{-\infty}^{\infty}\hat{a}\psi^*\phi\, dx = \langle\hat{a}\psi|\phi\rangle$$

$$\therefore \langle\hat{a}\psi|\phi\rangle = \langle\psi|\hat{a}^\dagger\phi\rangle \text{ proved.}$$

$\rightarrow$ Lets use what we proved in the box to get c_n & d_n.

So we have: $\langle \hat{a}^+\psi | \hat{a}^+\phi\rangle = \langle \hat{a}\hat{a}^+\psi | \phi\rangle$ and

$\langle \hat{a}\psi | \hat{a}\phi\rangle = \langle \hat{a}^+\hat{a}\psi | \phi\rangle$

By using ⊕' and $\hat{H}\psi = E\psi$ where $\hat{H}$ is represented in $\hat{a}$ & $\hat{a}^+$.

We can say.

$$\left[\hat{a}^+\hat{a}|\psi_n\rangle = n|\psi_n\rangle \quad \& \quad \hat{a}\hat{a}^+|\psi_n\rangle = (n+1)|\psi_n\rangle\right] \text{ — } \#'''$$

First lower then raise $\rightarrow$ leave n. First raise then lower $\rightarrow$ leave $(n+1)$

So, using #''

$$\underbrace{\langle \hat{a}^+\psi_n | \hat{a}^+\psi_n\rangle}_{= \langle \hat{a}\hat{a}^+\psi_n|\psi_n\rangle} = \langle c_n\psi_{n+1} | c_n\psi_{n+1}\rangle = |c_n|^2 \underbrace{\langle \psi_{n+1} | \psi_{n+1}\rangle}_{=1 \text{ (Normalized)}} \overset{\#'''}{=} (n+1)\underbrace{\langle \psi_n | \psi_n\rangle}_{=1}$$

$$\therefore |c_n|^2 = (n+1) \Rightarrow c_n = \sqrt{n+1}$$

$$\underbrace{\langle \hat{a}\psi_n | \hat{a}\psi_n\rangle}_{= \langle \hat{a}^+\hat{a}\psi_n|\psi_n\rangle} \overset{\#''}{=} \langle d_n\psi_{n-1} | d_n\psi_{n-1}\rangle = |d_n|^2 \underbrace{\langle \psi_{n-1} | \psi_{n-1}\rangle}_{=1} = n\underbrace{\langle \psi_n | \psi_n\rangle}_{=1}$$

$$\therefore |d_n|^2 = n \rightarrow d_n = \sqrt{n}.$$

$$\therefore \hat{a}^+|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle \quad \& \quad \hat{a}|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle$$

$$\rightarrow |\psi_1\rangle = \frac{\hat{a}^+|\psi_0\rangle}{\sqrt{1}}, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\hat{a}^+|\psi_1\rangle, \quad |\psi_3\rangle = \frac{1}{\sqrt{3}}\hat{a}^+|\psi_2\rangle \ \ldots$$

$$\therefore |\psi_n\rangle = \frac{1}{\sqrt{n}}(\hat{a}^+)^n|\psi_0\rangle$$