Problem 3 : Spin Problem

(3.a) Derive the Commutation relations

Comment

We basically derive the general angular momentum property $[L_i, L_j] = 2i\epsilon_{ijk}L_k$ for the case of $\frac{1}{2}$ spin operators.

Problem (i)

Derive

$$[\hat{\sigma_x}\hat{\sigma_y}] = 2i\hat{\sigma_z} \tag{1}$$

Solution

$$\begin{aligned}
\text{LHS} =&[\hat{\sigma_x}, \hat{\sigma_y}] \\
=&\hat{\sigma_x}\hat{\sigma_y} - \hat{\sigma_y}\hat{\sigma_x} \\
=&\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
=&\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\
=&2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\hat{\sigma_z} = \text{RHS}
\end{aligned}$$

Problem (ii)

Derive (Can get a little tricky)

$$[\hat{\sigma_z}, \hat{\sigma_x}] = 2i\hat{\sigma_y} \tag{2}$$

Solution

$$\begin{aligned}
\text{LHS} = [\hat{\sigma_z}, \hat{\sigma_x}] = \hat{\sigma_z}\hat{\sigma_x} - \hat{\sigma_x}\hat{\sigma_z} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \Big| * \frac{i}{i} \\
&= 2i\begin{bmatrix} 0 & \frac{1}{i} \\ \frac{i^2}{i} & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & \frac{1}{i}\frac{i}{i} \\ i & 0 \end{bmatrix} \\
&= 2i\begin{bmatrix} 0 & \frac{i}{-1} \\ i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 2i\hat{\sigma_y} \\
&= \text{RHS}
\end{aligned}$$

Problem (iii)

Derive

$$[\hat{\sigma_y}\hat{\sigma_z} - \hat{\sigma_z}\hat{\sigma_y}] = 2i\hat{\sigma_x} \tag{3}$$

Solution

$$\begin{aligned}
\text{LHS} &= [\hat{\sigma_y}\hat{\sigma_z} - \hat{\sigma_z}\hat{\sigma_y}] \\
&= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\hat{\sigma_x}
\end{aligned}$$

□

(3.b) Eigenvectors and Eigenvalues

Problem

Give the eigenstates of the Hamilton operator $\hat{H} = \epsilon\hat{\sigma}_z$ where $\epsilon \in \mathbb{R}$

Solution

Our Hamilton is defined as

$$\hat{H} = \epsilon\hat{\sigma}_z = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}$$

Being a diagonal matrix, the eigenvalues are just the diagonal components and the eigenvectors are the corresponding column vectors. Let λ_1, λ_2 be the eigenvalues for $\vec{v}_1, \vec{v}_2$ as the eigenvectors for $\hat{H}$, then

$$\lambda_1 = +\epsilon\,, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -\epsilon\,, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Comment

Maybe you have to derive them and not just use this fact about diagonal matrices. Ask the invigilator to confirm.
If the Hamiltonian was made out of $\hat{\sigma}_x$ or $\hat{\sigma}_y$, then you would have had to compute the eigenvalues and then go for the eigenvectors.

(3.c) Spin in arbitrary direction

Comment

This problem tests your fundamental knowledge about measurement in quantum mechanics. It is in the simplest setting : a two state system with spin as your observable.

Problem

[Buildup]
Consider the spin operator in an arbitrary direction $\vec{n} = (n_x, n_y, n_z)$ given by

$$\hat{S}_{\vec{n}} = \frac{1}{2}(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)$$

(hbar=1), where $\vec{n}$ is a unit vector.

[Given] Let Ψ be the eigenstate of $\hat{H}$ with the eigenvalue $+\epsilon$.

[Question] What is the probability for measuring $\hat{S}_{\vec{n}}$ to have the value $\frac{1}{2}$ in the state Ψ?

Solution

Using the [Question] part of the problem, we can easily construct the probability equation as follows :

$$\mathcal{P} = \left|\langle n;+|z;+\rangle\right|^2 \tag{4}$$

Where,

- $|n;+\rangle$ represents the state satisfying $\hat{S}_{\vec{n}}|n;+\rangle = \frac{1}{2}|n;+\rangle$
- $|z;+\rangle$ represents the state satisfying $\hat{H}|z;+\rangle = +\epsilon|z;+\rangle$.
  - From the previous sub-question (3.2) we know that

$$|z;+\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

Hence, our main goal is to derive an expression for $|n;+\rangle$. Let us first define the unit vector $\vec{n}$ using spherical coordinates

$$\vec{n} = \begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{bmatrix}$$

and

$$\hat{\vec{S}} = \begin{bmatrix}\hat{S}_x\\\hat{S}_y\\\hat{S}_z\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\hat{\sigma}_x\\\hat{\sigma}_y\\\hat{\sigma}_z\end{bmatrix}$$

We also know ,

$$
\begin{aligned}
\hat{S}_{\vec{n}} = \vec{n}\cdot\hat{\vec{S}} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{bmatrix} &= \frac{1}{2}(n_x\hat{\sigma_x} + n_y\hat{\sigma_y} + n_z\hat{\sigma_z}) \\
&= \frac{1}{2}\left(n_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + n_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \\
&= \frac{1}{2}\begin{bmatrix} n_z & n_x - i\,n_y \\ n_x + i\,n_y & -n_z \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\; e^{-i\phi} \\ \sin\theta\; e^{i\phi} & -\cos\theta \end{bmatrix} && \text{(Euler's formula)}
\end{aligned}
$$

Comment

The phases above, $e^{\pm i\phi}$ are pure phases and do not play a physical role in QM. If you don't know this, go back and try to understand why. Hint : It has something to do with the probabilistic interpretation of states. Pure phases do not contribute in the Born interpretation.

Now that we have an expression for $\hat{S}_{\vec{n}}$, we need to find the state $|n;+\rangle$ such that

$$
\hat{S}_{\vec{n}}\,|n;+\rangle = +\frac{1}{2}\,|n;+\rangle \tag{5}
$$

(One can check that the eigenvalues of $\hat{S}_{\vec{n}}$ are indeed $\pm\frac{1}{2}$)

We can define $|n;+\rangle$ in as a linear combination of the z states in the following way

$$
|n;+\rangle = c_1\,|z;+\rangle + c_2\,|z;-\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
$$

We can rearrange the (5) to get the typical equation we solve for eigenvectors.

$$
\begin{aligned}
\left(\hat{S}_{\vec{n}} - \hat{I}\frac{1}{2}\right)|n;+\rangle &= 0 \\
\left(\frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\; e^{-i\phi} \\ \sin\theta\; e^{i\phi} & -\cos\theta \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)|n;+\rangle &= 0 \\
\frac{1}{2}\begin{bmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= 0
\end{aligned}
$$

This equation gives us two equations

$$\frac{1}{2}[c_1(\cos\theta - 1) + c_2(\sin\theta\, e^{-i\phi})] = 0$$
$$\frac{1}{2}[c_1(\sin\theta\, e^{i\phi}) - c_2(\cos\theta + 1)] = 0$$

Solving the first one of them gives us

$$c_1(\cos\theta - 1) = -c_2(\sin\theta\, e^{-i\phi})$$
$$c_2 = \frac{c_1(1-\cos\theta)}{\sin\theta} e^{i\phi}$$

We won't solve the second equation because it gives us exactly the same relation between c_1 and c_2 . (You can definitely check for yourself!)

We can use some trigonometric identities to simplify the relation to the following form

$$c_2 = \left(e^{i\phi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right) c_1$$

We have the $|n;+\rangle$ vector in the z-state basis in the following form

$$|n;+\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ \left(e^{i\phi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right) c_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \left(e^{i\phi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right) \end{bmatrix}$$

We are almost there, one last step left. Can you guess? ...We gotta normalize the state. We start with the normalization condition

$$\langle n;+|n;+\rangle = 1$$
$$c_1^* c_1 + c_2^* c_2 = 1$$
$$|\ c_1\ |^2 + |\ c_2\ |^2 = 1$$
$$|\ c_1\ |^2 \left(1 + \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} \right) = 1$$
$$|\ c_1\ |^2 \left(\frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} \right) = 1$$
$$|\ c_1\ |^2 = \cos^2\frac{\theta}{2}$$

Taking the simplest form for c_1 we get

$$c_1 = \cos\frac{\theta}{2}, \quad c_2 = \sin\frac{\theta}{2} e^{i\phi}$$

Which FINALLY gives us the $|n;+\rangle$ state as follows

$$|n;+\rangle = \cos\frac{\theta}{2}\,|z;+\rangle + \sin\frac{\theta}{2} e^{i\phi}\,|z;-\rangle$$

Recalling that $|z;+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |z;-\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ we can also write the above equation as

$$|n;+\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

in the z-state basis.

Finally,we are in a state to calculate the probability. Recalling the first equation we wrote in this solution for the probability i.e. eq.(4)

$$\begin{aligned}
\mathcal{P} &= \left|\langle n;+|z;+\rangle\right|^2 \\
&= \left|\begin{bmatrix}(\cos\frac{\theta}{2})^* & (\sin\frac{\theta}{2}e^{i\phi})^*\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 \\
&= \left|\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 \\
&= \left|\cos\frac{\theta}{2} + 0\right|^2 = \cos^2\frac{\theta}{2}
\end{aligned}$$

Therefore, if you have a state prepared which corresponds to the $+\frac{1}{2}$ eigenvalue of the $\hat{S}_z$ operator i.e the $|z;+\rangle$ state

. . . then . . .

the probability that one can measure $\frac{1}{2}$ for the $\hat{S}_{\vec{n}}$ observable depends on the polar angle θ and the exact probability is given by $\cos^2\frac{\theta}{2}$

Comment

Check if this makes sense with some obvious examples

- Example 1 : $\theta_1 = 0$.

  For this we have $|n;+\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ which is the same as being exactly in the state $|z;+\rangle$.

  The probability is

  $$\mathcal{P}_{\theta_1} = \cos^2\frac{\theta_1}{2} = \cos^2 0 = 1$$

  which makes perfect sense as you are exactly in the same eigenstate and being at a polar angle of $\theta = 0$ corresponds to being in $|z;+\rangle$

- Example 2: $\theta_2 = \pi$

  For this we have $|n;+\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ which is the same as being exactly in the state $|z;-\rangle$.

  The probability is

  $$\mathcal{P}_{\theta_2} = \cos^2\frac{\theta_2}{2} = \cos^2\frac{\pi}{2} = 0$$

  which again makes perfect sense as being at a polar angle $\theta_2 = \pi$ corresponds to being in $|z;-\rangle$ state.

(3.d) Time evolution of an operator

Comment

This is how computations in the Heisenberg picture are done. Operators are time dependent and the states are steady, compared to the Schrodinger picture where it is vice versa.

Problem

Derive the time evolution

$$\hat{\sigma}_x(t) = \cos(\omega t)\hat{\sigma}_x - \sin(\omega t)\hat{\sigma}_y$$

(I believe there is typo in the question in the Mock 2017, there is an extra factor of i in front of the second term in the RHS)
We have $\omega = \frac{2\epsilon}{\hbar}$ which has the dimension of frequency.

Solution

In order to derive the **time evolution of an operator** of the system, we need to recall what is the Hamiltonian for this problem

$$\hat{H} = \epsilon\,\hat{\sigma}_z = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

Let the operator that we want to evolve with respect to time be denoted by $\hat{A}$. Which in this case is

$$\hat{A} = \hat{\sigma}_x$$

Let the time evolved operator be denoted by $\hat{A}_t$.
Using the Heisenberg's picture, the formula that we want to use to get $\hat{A}_t$ from $\hat{H}, \hat{A}$ is by sandwiching the $\hat{A}$ between these two operators made from $\hat{H}$

$$\hat{A}_t = \exp\left(i\left(\frac{\hat{H}}{\hbar}\right)t\right) \quad \hat{A} \quad \exp\left(-i\left(\frac{\hat{H}}{\hbar}\right)t\right)$$

Let $\hbar = 1$ and by substituting the appropriate $\hat{A}, \hat{H}, \hat{A}_t$ for this problem we get (and I am going to ignore the hats on the sigmas probably at some point),

$$\hat{\sigma}_x(t) = \exp(i(\epsilon\,\hat{\sigma}_z)t) \quad \hat{\sigma}_x \quad \exp(-i(\epsilon\,\hat{\sigma}_z)t) \tag{7}$$

We get the time evolved operator by doing the computation on the RHS in the equation above. There are two methods to do this :

Method 1

This is the more general method for computing these kinds of equations and works for many similar computations. (Method 2, might have some limitations)

The goal of this method, is to form a differential equation from eq.(7) for which we can easily guess the solution. (Your Harmonic oscillator bells should start ringing now)
Algorithm for this method :

- The eq.(7) is mainly a function of time. For notational convenience we will redefine it as
$$f(t) = \exp(i(\epsilon\,\hat{\sigma}_z)t) \quad \hat{\sigma}_x \quad \exp(-i(\epsilon\,\hat{\sigma}_z)t)$$
Where we can see $f(0) = \hat{\sigma}_x$.

- We have to form a differential equation from this, so let us take a derivative w.r.t t
$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\exp(i(\epsilon\,\hat{\sigma}_z)t) \quad \hat{\sigma}_x \quad \exp(-i(\epsilon\,\hat{\sigma}_z)t)\big)$$
When you do this differentiation, you have to be very careful with the order of the terms as you are dealing with operators that do not commute. Also, when you use the chain rule, the outer derivative goes first and then goes the inner ones i.e. $\frac{\mathrm{d}\phi(x(t))}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \neq \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}x}$ when you are dealing with functions of operators.

  Now let us compute this

  – First use the product rule to get this form
  $$f'(t) = \frac{\mathrm{d}\exp\big(i(\epsilon\hat{\sigma}_z)t\big)}{\mathrm{d}t}\hat{\sigma}_x \exp\big(-i(\epsilon\hat{\sigma}_z)t\big) + \exp\big(i(\epsilon\hat{\sigma}_z)t\big)\hat{\sigma}_x \frac{\mathrm{d}\exp\big(-i(\epsilon\hat{\sigma}_z)t\big)}{\mathrm{d}t}$$
  – Now use the comment about the chain rule to evaluate the derivatives
  $$f'(t) = \underbrace{\exp\big(i(\epsilon\hat{\sigma}_z)t\big)}_{\text{Outer}}\ \underbrace{(i\epsilon\hat{\sigma}_z)}_{\text{Inner}}\hat{\sigma}_x \exp\big(-i(\epsilon\hat{\sigma}_z)t\big) + \exp\big(i(\epsilon\hat{\sigma}_z)t\big)\hat{\sigma}_x \underbrace{\exp\big(-i(\epsilon\hat{\sigma}_z)t\big)}_{\text{Outer}}\underbrace{(-i\epsilon\hat{\sigma}_z)}_{\text{Inner}}$$
  Where outer and inner refers to the outer and inner derivatives from the chain rule.
  – If you look carefully, in both the derivative expressions, the inner is the **operator** and outer is the **function of the same operator**. Hence, they commute and we can interchange the **inner** and **outer** expressions in both the terms in the above equation. In this step, we will only interchange the places of the inner and outer expressions in the second term. And after that, just pull out all the constants in front of each terms.
  $$f'(t) = i\epsilon\ e^{i(\epsilon\hat{\sigma}_z)t}(\hat{\sigma}_z\hat{\sigma}_x)e^{-i(\epsilon\hat{\sigma}_z)t} - i\epsilon\ e^{i(\epsilon\hat{\sigma}_z)t}(\hat{\sigma}_x\hat{\sigma}_z)e^{-i(\epsilon\hat{\sigma}_z)t}$$
  (Take a minute to see where this expression comes from and why were we able to interchange the inner and outer expressions in the second term on the RHS)
  – This expression has a very special form, in order to see that form let us redefine some expressions as follows
  $$k = i\epsilon, \quad g_p(t) = e^{i(\epsilon\hat{\sigma}_z)t}, \quad g_m(t) = e^{-i(\epsilon\hat{\sigma}_z)t}$$
  Using these we have (we will assume $g_p(t)$ and $g_m(t)$ are functions of t and will write them as g_p and g_m unless we need to explicitly mention otherwise)
  $$f'(t) = k(g_p(\hat{\sigma}_z\hat{\sigma}_x)g_m - g_p(\hat{\sigma}_x\hat{\sigma}_z)g_m)$$
  Now, I can pull g_p from the **left** hand side of the bracket and g_m from the **right** hand side of the bracket. Doing this I get
  $$\begin{aligned} f'(t) &= k\ g_p(\hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z)g_m \\ &= k\ g_p[\hat{\sigma}_z, \hat{\sigma}_x]g_m \\ &= kg_p(2i\hat{\sigma}_y)g_m \\ &= 2ki\ g_p(\hat{\sigma}_y)g_m, \quad (2ki = 2i^2\epsilon = -2\epsilon) \\ &= -2\epsilon\ g_p(\hat{\sigma}_y)g_m \end{aligned}$$

As a side note we can keep

$$f'(0) = -2\epsilon\hat{\sigma}_y$$

as $g_p(t=0) = g_m(t=0) = 1$.

- Let us compare $f(t), f'(t)$ side-by-side to see if we can relate them in some way to form a differential equation.

$$f(t) = g_p\hat{\sigma}_x g_m, \quad f'(t) = -2\epsilon g_p(\hat{\sigma}_y)g_m$$

There is no way to relate these two expressions to form a differential equation. So, we go on our hunt to find the second derivative.

- Computing the second derivative similar to the first one you get to this point

$$\begin{aligned} f''(t) &= -2\epsilon\frac{\mathrm{d}(g_p\hat{\sigma}_y g_m)}{\mathrm{d}t} \\ &= -2\epsilon(i\epsilon)(g_p\big[\hat{\sigma}_z,\hat{\sigma}_y\big]g_m), \quad \big[\hat{\sigma}_z,\hat{\sigma}_y\big] = -2i\hat{\sigma}_x \\ &= -2\epsilon^2 i(-2i)(g_p\,\hat{\sigma}_x g_m) \\ &= 4\epsilon^2 i^2(g_p\,\hat{\sigma}_x g_m) \\ &= -4\epsilon^2(g_p\hat{\sigma}_x g_m) \end{aligned}$$

(Try to compute this on your own, if not, ask around if someone was able to, if not, then contact me :P)

- Let us compare all the two derivatives of $f\,(t)$ with itself side-by-side to see if we can spot a differential equation using these expressions

$$f(t) = (g_p\hat{\sigma}_x g_m), \quad f'(t) = -2\epsilon(g_p\hat{\sigma}_y g_m), \quad f''(t) = -4\epsilon(g_p\,\hat{\sigma}_x g_m)$$

We can see a clear mathematical relation between $f(t)$ and $f''(t)$,

$$f''(t) = -4\epsilon^2\underbrace{(g_p\,\hat{\sigma}_x g_m)}_{=f(t)} = -4\epsilon^2 f(t)$$

$$f''(t) = -4\epsilon^2 f(t) \qquad (\maltese)$$

$$f(0) = \hat{\sigma}_x, \quad f'(0) = -2\epsilon\hat{\sigma}_y \qquad \text{(IC)}$$

- We have a second order differential equation : eq.(✠), and two initial conditions (IC). The eq (✠) is a Harmonic oscillator equation where we can already see that $\omega = 2\epsilon$ as in the question.
We make the obvious harmonic oscillator ansatz ,

$$\begin{aligned} f(t) &= \alpha\sin(\omega t) + \beta\cos(\omega t) \\ f'(t) &= \omega\alpha\cos(\omega t) - \omega\beta\sin(\omega t) \end{aligned}$$

Now use the initial conditions to find α, β.

$$\begin{aligned} f(0) &= \beta = \hat{\sigma}_x \\ f'(0) &= \omega\alpha = -2\epsilon\hat{\sigma}_y \end{aligned}$$

solving them gives us

$$\alpha = -\frac{2\epsilon}{\omega}\hat{\sigma}_y = -\frac{2\epsilon}{2\epsilon}\hat{\sigma}_y = -\hat{\sigma}_y, \quad \beta = \hat{\sigma}_x$$

- Plugging the α, β into our ansatz and recalling that $f(t) = \hat{\sigma}_x(t)$ we get

$$\hat{\sigma}_x(t) = f(t) = -\hat{\sigma}_y \sin(\omega t) + \hat{\sigma}_x \cos(\omega t)$$

  where $\omega = 2\epsilon$

Method 2

To be addded

$\square$

(3.e) Derive the uncertainty for $\hat{S}_x$ and $\hat{S}_y$ in the Ψ state

Problem 3.e

Derive an uncertainty relation between $\hat{S}_x$ and $\hat{S}_y$ for the state Ψ

Solution

We have the two operators defined as (with $\hbar$ as it is)

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x, \quad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y$$

The uncertainty relation is defined for any two operators $\hat{A}, \hat{B}$ as follows

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2} \left| \frac{1}{i}\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle_\Psi \right|$$

Using this for our operators being $\hat{A} = \hat{S}_x$, $\hat{B} = \hat{S}_y$ and $\left[\hat{S}_x, \hat{S}_y\right] = 2i\hat{S}_z$ we get (Also, recall that $\hat{S}_\alpha = \frac{\hbar}{2}\hat{\sigma}_\alpha$ where $\alpha \in \{x, y, z\}$)

$$\Delta\hat{S}_x\Delta\hat{S}_y \geq \frac{1}{2} \left| \frac{1}{i}\left\langle \left[\hat{S}_x, \hat{S}_y\right] \right\rangle_\Psi \right| = \frac{1}{2} \left| \frac{1}{i}\left\langle 2i\hat{S}_z \right\rangle_\Psi \right|$$

One can take constants out of the expectation brackets to get

$$\Delta\hat{S}_x\Delta\hat{S}_y \geq \frac{1}{2} \left| \frac{1}{i}(2i)\left\langle \hat{S}_z \right\rangle_\Psi \right| = \frac{1}{2} \left| 2\left\langle \hat{S}_z \right\rangle_\Psi \right|$$

Now taking the constants out of the absolute value brackets

$$\Delta\hat{S}_x\Delta\hat{S}_y \geq \left| \left\langle \hat{S}_z \right\rangle_\Psi \right|$$

The expectation value of $\hat{S}_z$ in the state $\Psi = \begin{bmatrix} +\epsilon \\ 0 \end{bmatrix}$ is the eigenvalue $+\frac{\hbar}{2}$ (because Ψ is a pure eignvector of $\hat{S}_z$).
Therefore we get the uncertainty relation as

$$\Delta\hat{S}_x\Delta\hat{S}_y \geq \frac{\hbar}{2}$$

(3.f) Time invariance of the Uncertainty principle

Problem 3.f

Is the uncertainty preserved under time evolution. (Explain your answer)

Solution

(We did this exact derivation in one of our lectures and we know that uncertainty is indeed preserved under time evolution)
There are two ways of looking about this problem :

1. Schrodinger picture (States are time dependent and Operators are static)

$$\left|\Psi(t)\right\rangle , \quad \hat{\mathcal{O}}$$

2. Heisenberg picture (States are static and the operators are time dependent)

$$\left|\Psi\right\rangle , \quad \hat{\mathcal{O}}(t)$$

We will write down the equation which connects both these pictures, where $\Psi(t) = \Psi_t$ are the states in the Schrodinger picture and $\Psi = \Psi_0$ are the states in the Heisenberg picture.

$$\left\langle\Psi_t\right| \hat{\mathcal{O}}^2 \left|\Psi_t\right\rangle$$