Problem 1 : Creation/Annihiliation operators

Given for all parts

- Assume we have an operator $\hat{a}$ such that $[\hat{a}, \hat{a}^\dagger] = \hat{I}$
- Let $\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{I}$

Problem (a)

Suppose Ψ_0 is a state such that $\hat{a}\Psi_0 = 0$. Show how to obtain normalized eigenstates of $\hat{H}$ from this state.
(I will use the BraKet notation for simplicity : $\hat{a}\,|\Psi_0\rangle$)

Solution

We are given $\hat{a}\,|\Psi_0\rangle = 0$ and we want to find the normalized eigenstates of $\hat{H}$. Let us start by checking how $\hat{H}$ acts on $|\Psi_0\rangle$

$$\begin{aligned}\hat{H}\,|\Psi_0\rangle &= \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{I}\right)|\Psi_0\rangle \\ &= \hat{a}^\dagger \underbrace{\hat{a}\,|\Psi_0\rangle}_{=0\text{ (Given)}} + \frac{1}{2}\hat{I}\,|\Psi_0\rangle \\ &= \frac{1}{2}\,|\Psi_0\rangle\end{aligned}$$

Hence, $|\Psi_0\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue $\frac{1}{2}$.

Look at the Hamiltonian carefully

$$\hat{H} = \underbrace{\hat{a}^\dagger \hat{a}}_{=\hat{N}} + \frac{1}{2}\hat{I}$$

The eigenstates of $\hat{N}$ and $\hat{H}$ will be equivalent. Eigenvalues will differ by $\frac{1}{2}$ due to the second term $\frac{1}{2}\hat{I}$. Thus, finding the eigenstates of $\hat{H}$ is equivalent to finding the eigenstates of $\hat{N}$ which is called the *Number operator*.

Let's assume that $|\psi_n\rangle$ are the eigenstates of the number operator $\hat{N}$ with the eigenvalues λ_n (Then, $\lambda_n + \frac{1}{2}$ would be the eigenvalue for the $\hat{H}$ with the same eigenstate $|\psi_n\rangle$),

$$\hat{N}\,|\psi_n\rangle = \lambda_n\,|\psi_n\rangle \quad , \quad \hat{H}\,|\psi_n\rangle = \left(\lambda_n + \tfrac{1}{2}\right)|\psi_n\rangle$$

Now, we will do some magic using the $\hat{N}$ along with $\hat{a}^\dagger, \hat{a}$ operators. The magic is due to the following commutation relations which we can use later to prove some powerful statements.

- Commutation relation between $\hat{N}$ and $\hat{a}$

$$\left[\hat{N}, \hat{a}\right] = \left[\hat{a}^\dagger \hat{a}, \hat{a}\right] = \left[\hat{a}, \hat{a}^\dagger\right]\hat{a} + \hat{a}^\dagger \underbrace{[\hat{a}, \hat{a}]}_{=0} = \left[\hat{a}, \hat{a}^\dagger\right]\hat{a} = -\hat{I}\hat{a} = -\hat{a}$$

- Commutation relation between $\hat{N}$ and $\hat{a}^\dagger$

$$\left[\hat{N}, \hat{a}^\dagger\right] = \left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger\right] = \underbrace{\left[\hat{a}^\dagger, \hat{a}^\dagger\right]}_{=0} \hat{a} + \hat{a}^\dagger \left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger \left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger \hat{I} = \hat{I}\hat{a}^\dagger = \hat{a}^\dagger$$

$$\left[\hat{N}, \hat{a}\right] = -\hat{a} \quad \text{and} \quad \left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$$

We know that $|\psi_n\rangle$ are the eigenstates of the operator $\hat{N}$. We want to check two things,

1. The state $\hat{a}^\dagger |\psi_n\rangle$ is also an eigenstate of $\hat{N}$ with the eigenvalue $(\lambda_n + 1)$

$$\begin{aligned} \hat{N}\left(\hat{a}^\dagger |\psi_n\rangle\right) &= \left(\hat{N}\hat{a}^\dagger\right) |\psi_n\rangle = \left(\hat{a}^\dagger \hat{N} + \hat{a}^\dagger\right) |\psi_n\rangle = \left(\hat{a}^\dagger \hat{N} |\psi_n\rangle + \hat{a}^\dagger |\psi_n\rangle\right) \\ &= \left(\lambda_n \hat{a}^\dagger |\psi_n\rangle + \hat{a}^\dagger |\psi_n\rangle\right) = (\lambda_n + 1)\hat{a}^\dagger |\psi_n\rangle \end{aligned}$$

   Hence proved that $\hat{a}^\dagger |\psi_n\rangle$ is also an eigenstate of $\hat{N}$ with an eigenvalue $\lambda_n + 1$. (This is where $\hat{a}^\dagger$ gets the name *creation operator*)

2. The state $\hat{a} |\psi_n\rangle$ is also an eigenstate of $\hat{N}$ with the eigenvalue $\lambda_n - 1$

$$\begin{aligned} \hat{N}\left(\hat{a} |\psi_n\rangle\right) &= \left(\hat{N}\hat{a}\right) |\psi_n\rangle = \left(\hat{a}\hat{N} - \hat{a}\right) |\psi_n\rangle = \left(\hat{a}\hat{N} |\psi_n\rangle - \hat{a} |\psi_n\rangle\right) \\ &= \left(\lambda_n \hat{a} |\psi_n\rangle - \hat{a} |\psi_n\rangle\right) = (\lambda_n - 1)\hat{a} |\psi_n\rangle \end{aligned}$$

   Hence proved that $\hat{a} |\psi_n\rangle$ is an eigenstate of $\hat{N}$ with an eigenvalue $\lambda_n - 1$. (This where $\hat{a}$ gets the name *destruction/annihiliation operator*)

$$\hat{N}\left(\hat{a}^\dagger |\psi_n\rangle\right) = (\lambda_n + 1) |\psi_{n+1}\rangle, \quad \hat{N}\left(\hat{a} |\psi_n\rangle\right) = (\lambda_n - 1) |\psi_{n-1}\rangle,$$

Using the two results from above, we can make the following conclusions,

1. $\hat{N}(\hat{a}^\dagger |\psi_n\rangle) = (\lambda_n + 1) |\psi_n\rangle$ implies

$$\begin{aligned} \hat{a}^\dagger |\psi_n\rangle &\propto |\psi_{n+1}\rangle \\ &= c_+ |\psi_{n+1}\rangle \end{aligned}$$

   (not '= $|\psi_{n+1}\rangle$' because $\hat{a}^\dagger |\psi_n\rangle$ need not be normalized)

2. $\hat{N}(\hat{a} |\psi_n\rangle) = (\lambda_n - 1) |\psi_n\rangle$ implies that

$$\begin{aligned} \hat{a} |\psi_n\rangle &\propto |\psi_{n-1}\rangle \\ &= c_- |\psi_{n-1}\rangle \end{aligned}$$

   (not '=$|\psi_{n-1}\rangle$' because $\hat{a} |\psi_n\rangle$ need not be normalized)

$$\hat{a}^\dagger |\psi_n\rangle = c_+ |\psi_{n+1}\rangle, \quad \hat{a} |\psi_n\rangle = c_- |\psi_{n-1}\rangle$$

Now, we will use the normalization condition to find the proportionality constant for the two equations above

1.

$$\begin{aligned}
\left\langle \hat{a}^\dagger \psi_n \middle| \hat{a}^\dagger \psi_n \right\rangle &= \left\langle \psi_n \middle| \hat{a}\hat{a}^\dagger \psi_n \right\rangle \\
&= \left\langle \psi_n \middle| \left(\hat{a}^\dagger \hat{a} + \hat{I}\right) \psi_n \right\rangle \\
&= \left\langle \psi_n \middle| \left(\hat{N} + \hat{I}\right) \psi_n \right\rangle \\
&= \langle \psi_n | (\lambda_n + 1) \psi_n \rangle \\
&= (\lambda_n + 1) \underbrace{\langle \psi_n | \psi_n \rangle}_{=1} \\
\left\langle \hat{a}^\dagger \psi_n \middle| \hat{a}^\dagger \psi_n \right\rangle &= (\lambda_n + 1) \\
\underbrace{c_+ c_+^*}_{\|c_+\|^2} \underbrace{\langle \psi_{n+1} | \psi_{n+1} \rangle}_{=1} &= (\lambda_n + 1) \\
\|c_+\| &= \sqrt{\lambda_n + 1}
\end{aligned}$$

2. Similarly,

$$\begin{aligned}
\langle \hat{a}\psi_n | \hat{a}\psi_n \rangle &= \left\langle \psi_n \middle| \hat{a}^\dagger \hat{a} \psi_n \right\rangle \\
&= \left\langle \psi_n \middle| \hat{N} \psi_n \right\rangle \\
\underbrace{c_- c_-^*}_{\|c_-\|^2} \underbrace{\langle \psi_{n-1} | \psi_{n-1} \rangle}_{=1} &= \lambda_n \underbrace{\langle \psi_n | \psi_n \rangle}_{=1} \\
\|c_-\| &= \sqrt{\lambda_n}
\end{aligned}$$

Therefore, we have the following two crucial results

$$\hat{a}^\dagger |\psi_n\rangle = \sqrt{\lambda_n + 1}\, |\psi_{n+1}\rangle, \quad \hat{a} |\psi_n\rangle = \sqrt{\lambda_n}\, |\psi_{n-1}\rangle$$

Which can also be rearranged as follows

$$|\psi_{n+1}\rangle = \frac{\hat{a}^\dagger |\psi_n\rangle}{\sqrt{\lambda_n + 1}}, \quad |\psi_{n-1}\rangle = \frac{\hat{a} |\psi_n\rangle}{\sqrt{\lambda_n}}$$

We can go a step ahead and act with $\hat{a}^\dagger$ on $|\psi_{n-1}\rangle$ s

$$\begin{aligned}
\hat{a}^\dagger |\psi_{n-1}\rangle &= \frac{\hat{a}^\dagger \left(\hat{a} |\psi_n\rangle\right)}{\sqrt{\lambda_n}} \\
&= \frac{\hat{N} |\psi_n\rangle}{\sqrt{\lambda_n}} = \frac{\lambda_n}{\sqrt{\lambda_n}} |\psi_n\rangle = \sqrt{\lambda_n}\, |\psi_n\rangle
\end{aligned}$$

Which can be rearranged as

$$|\psi_n\rangle = \frac{\hat{a}^\dagger |\psi_{n-1}\rangle}{\sqrt{\lambda_n}}$$

We will use the last formula in the previous box to substitute $|\psi_n\rangle$ in the formula for $|\psi_{n+1}\rangle$ (2nd line in the previous box)

$$|\psi_{n+1}\rangle = \frac{\hat{a}^\dagger |\psi_n\rangle}{\sqrt{\lambda_n + 1}} = \frac{\hat{a}^\dagger}{\sqrt{\lambda_n + 1}} \cdot \frac{\hat{a}^\dagger |\psi_{n-1}\rangle}{\sqrt{\lambda_n}} = \ldots = \frac{(\hat{a}^\dagger)^{n+1} |\psi_0\rangle}{\sqrt{(\lambda_n + 1)!}}$$

Going from $n+1 \to k$, we also go from $\lambda_n + 1 \to \lambda_k$ we can write

$$|\psi_k\rangle = \frac{(\hat{a}^\dagger)^k |\psi_0\rangle}{\sqrt{(\lambda_k)!}}$$

(Where we can replace k by a n wlog)
This means that, if we know $|\psi_0\rangle$ then you can find any $|\psi_n\rangle$.

Let's calculate the lowest possible eigenstate $|\psi_0\rangle$

$$\begin{aligned} \hat{a} |\psi_0\rangle &= 0 \\ (\hat{x} + i\hat{p}) |\psi_0\rangle &= 0 \\ \left(\hat{x} |\psi_0\rangle + \frac{i}{i} \frac{\mathrm{d} |\psi_0\rangle}{\mathrm{d}x} \right) &= 0 \\ \frac{\mathrm{d} |\psi_0\rangle}{|\psi_0\rangle} &= -x\mathrm{d}x \end{aligned}$$

Integrate both sides

$$\ln |\psi_0\rangle = -\frac{x^2}{2} + \mathcal{N}$$

$$|\psi_0\rangle = N \exp\left(-\frac{x^2}{2}\right)$$

Normalize this to get $N = \frac{1}{(\pi)^{\frac{1}{4}}}$
Therefore,

$$|\psi_0\rangle = \frac{1}{(\pi)^{\frac{1}{4}}} e^{-\frac{x^2}{2}}$$

Comment

A quick check that the expectation values of this number operator are positive,

$$\left\langle \hat{N} \right\rangle_{|\phi_n\rangle} = \langle\phi_n| \hat{N} |\phi_n\rangle = \alpha_n \langle\phi_n|\phi_n\rangle$$
$$= \langle\phi_n| \hat{a}^\dagger \hat{a} |\phi_n\rangle$$

We know that

$$\text{if we define} \quad \hat{a}|\phi_n\rangle = |\gamma_n\rangle \quad \text{then} \quad \langle\phi_n| \hat{a}^\dagger = \langle\gamma_n|$$

Therefore from this equality,

$$\langle\phi_n| \hat{a}^\dagger \hat{a} |\phi_n\rangle = \alpha_n \langle\phi_n|\phi_n\rangle$$

We can change the LHS using the definition right above and get,

$$\langle\gamma_n|\gamma_n\rangle = \alpha_n \langle\phi_n|\phi_n\rangle$$

By definition,

$$\langle\gamma_n|\gamma_n\rangle \geq 0 \quad \text{and} \quad \langle\phi_n|\phi_n\rangle \geq 0$$

which implies

$$\alpha_n \geq 0$$