Schrödinger & Heisenberg Pictures / Ehrenfest th^m

→ $\hat{H}$ → Time independent,

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$$

Heisenberg Operators

$$\hat{A}_H = U^\dagger(t,0)\,\hat{A}_s\,U(t,0)$$

1) At $t=0$, $\hat{A}_H(0) = \hat{A}_S$

$$\hat{1}_H = U^\dagger(t,0)\,1_s\,U(t,0) = \cancel{1_s}\,\hat{1}_s$$

2) $\hat{C}_S = \hat{A}_S\hat{B}_S \rightarrow \hat{C}_H(t) = \hat{A}_H(t)\hat{B}_H(t)$

3) $[\hat{A}_s,\hat{B}_s] = \hat{C}_s \longrightarrow [\hat{A}_H(t), \hat{B}_H(t)] = \hat{C}_H(t)$

4) $\langle\hat{A}_s\rangle = \langle\hat{A}_H(t)\rangle$

$$\Rightarrow \boxed{\langle\Psi,t|\hat{A}_s|\Psi,t\rangle = \langle\Psi,0|\hat{A}_H(t)|\Psi,0\rangle}$$

$$i\hbar\frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\left(\frac{\partial\hat{A}_s(t)}{\partial t}\right)_H$$

$\hat{A}_S$ without ~~tim~~ dep time →

$$i\hbar\frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H(t)]$$

$$\Rightarrow i\hbar\frac{d}{dt}\langle\hat{A}_H(t)\rangle = \langle[\hat{A}_H(t), \hat{H}_H(t)]\rangle$$

$$i\hbar\frac{d}{dt}\langle\hat{A}_s(t)\rangle = \langle[\hat{A}_s(t), \hat{H}_s(t)]\rangle$$

Example 1 Harmonic Oscillator.

$$\frac{d}{dt} A_H(t) = \frac{1}{i\hbar}\left[A_H(t), H_H(t)\right]$$

$$H_S = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\downarrow$$

$$H_H = \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2(t)$$

If $H_S(\hat{p}, \hat{x}; t)$ then $H_H(p_H(t), \hat{x}_H(t); t)$

$\rightarrow$ Consider $\hat{x}_S$ & $\hat{p}_S$ & use them in Heisenberg eqn of motion

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}\left[\hat{A}_H(t), \hat{H}_H(t)\right]$$

for $\hat{x}_H(t)$

$$\frac{d}{dt}\hat{x}_H(t) = \frac{1}{i\hbar}\left[\hat{x}_H(t), \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\hat{x}_H^2\omega^2\right]$$

$$= \frac{1}{i\hbar}\left(\left[\hat{x}_H(t), \frac{\hat{p}_H^2(t)}{2m}\right] + \underbrace{\left[\hat{x}_H(t), \frac{1}{2}m\hat{x}_H^2\omega\right]}_{0}\right)$$

$$= \cancel{\frac{1}{2mi\hbar}} \quad [A, BC] = [A,B]\hat{C} + \hat{B}[A,C]$$

$$= \frac{1}{2mi\hbar}\left[\hat{x}_H(t), \hat{p}_H(t)\hat{p}_H(t)\right]$$

$$= \frac{1}{2mi\hbar}\left(\underbrace{[\hat{x}_H(t), \hat{p}_H(t)]}_{i\hbar}\hat{p}_H(t) + \hat{p}_H(t)[\hat{x}_H(t), \hat{p}_H(t)]\right)$$

$$= \frac{2i\hbar}{2mi\hbar}\hat{p}_H(t) = \frac{\hat{p}_H(t)}{m}$$

$$\Rightarrow \frac{d}{dt}\hat{x}_H(t) = \frac{\hat{p}_H(t)}{m} \quad (*)$$

Eh

Ehrenfest thm

$$m\frac{d}{dt}\langle x\rangle = \langle p_H\rangle$$

For $\hat{p}$

$$\frac{d}{dt}\hat{p}_H(t) = \frac{1}{i\hbar}\left[\hat{p}_H(t), \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right]$$

$$[A, BC] = [A, B]\hat{C} + \hat{B}[A, \hat{C}]$$

$$= \frac{m\omega^2}{2i\hbar}\left[\hat{p}_H(t), \hat{x}^2\right]$$

$$= \frac{m\omega^2}{2i\hbar}\left(\underbrace{[\hat{p}_H(t), \hat{x}_H]}_{-i\hbar}\hat{x}_H + \hat{x}_H\underbrace{[\hat{p}_H, \hat{x}_H]}_{-i\hbar}\right)$$

$$= \frac{-2i\hbar}{2i\hbar} m\omega^2 \hat{x}_H$$

$$\therefore \frac{d}{dt}\hat{p}_H(t) = -m\omega^2\hat{x}_H \quad (*)^2$$

$$\rightarrow \frac{d\langle p\rangle}{dt} = -\nabla V = -\frac{d\langle V(x)\rangle}{dx}$$

$(*)^1 \rightarrow$

$$\frac{d^2}{dt^2}\hat{x}_H(t) = \frac{d}{dt}\frac{\hat{p}_H(t)}{m} = -\frac{m\omega^2\hat{x}}{m}$$

$$\Rightarrow \frac{d^2}{dt^2}\hat{x}_H(t) = -\omega^2\hat{x}_H(t)$$

$\hat{A}, \hat{B}$ → Time independent op defined by initial condition.

$$\therefore x_H(t) = \hat{A}\cos(\omega t) + \hat{B}\sin(\omega t)$$

$$p_H(t) = -m\omega\hat{A}\sin\omega t + m\omega\hat{B}\cos(\omega t)$$

$$\boxed{\begin{matrix}\hat{x}_H(0) = \hat{x}_s \\ \hat{p}_H(0) = \hat{p}_s\end{matrix}} \quad \begin{matrix}\Rightarrow \hat{x}_s = \hat{A} \\ \Rightarrow \hat{p}_s = \omega\hat{B}m \Rightarrow \hat{B} = \dfrac{\hat{p}_s}{\omega m}\end{matrix}$$

$$\therefore\ \hat{x}_H = \hat{x}_S \cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t$$

$\cancel{\hat{p}_P} =$

$$\hat{p}_H = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t.$$

Creation & Annihilation operators.

→ What are the Heisenberg operators corresponding to simple harmonic oscillator creation & annihilation operator.

$$\hat{a}_S \Rightarrow \hat{a}_H(t)$$
$$\hat{a}^+_S \Rightarrow \hat{a}^+_H(t)$$

S.H.O Hamiltonian → Time independent

⇒ We can use $\hat{A}_H(t) = \hat{U}^+(t,0)\hat{A}_S\hat{U}(t,0)$

$$= e^{+\frac{i}{\hbar}Ht}\hat{A}_S e^{-\frac{i}{\hbar}Ht}$$

$$\hat{H} = \hbar\omega(\underbrace{\hat{a}^+\hat{a}}_{\hat{N}} + \tfrac{1}{2})$$

Additive constant has no effect on the commutator.

How to evaluate $\hat{a}_H(t)$?

$$\hat{a}_H(t) = e^{+\frac{i}{\hbar}\hat{N}t}\hat{a}_S e^{-\frac{i}{\hbar}\hat{N}t} = e^{i\omega Nt}\hat{a}$$

$H = (\hat{a}^+\hat{a})\hbar\omega$

$$\frac{d}{dt}\hat{a}_H(t) = \omega i \cancel{\hbar} e^{i\cancel{\hbar}\hat{N}t\omega}[\hat{N},\hat{a}]e^{-i\hat{N}t\omega}$$

$$= \omega i\, e^{i\omega\hat{N}t}[\hat{N},\hat{a}]e^{-i\omega\hat{N}t}$$

$$= -\omega i\, e^{i\omega\hat{N}t}\hat{a}_S e^{-i\omega\hat{N}t}.$$

$$\therefore \frac{d}{dt}\hat{a}_H(t) = -\hat{a}_H(t)\, i\omega$$

$\hat{a}(t=0) = \hat{a}$

Soln: $\hat{a}_H(t) = \hat{a}e^{-i\omega t}$

$$\hat{a}^+_H(t) = \hat{a}^+e^{i\omega t}.$$