Special Relativity

A crash course of the main results that one should have seen before doing GR

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Contents

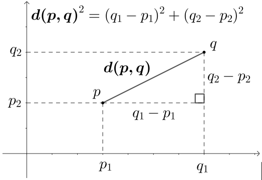
Metric

- What is a **metric**? What is it's purpose? (Also called as the *line element* - especially in Physics literature)
  - The term metric comes from "metric space". Metric space is a very basic structure that we can put on a set.

Metric space

- Metric space : set → together with a *metric* on the set.
  - The metric → function → defines a concept of distance → between any two members of the set, which are usually called points.
- Formal definition : A metric set is an ordered pair (M, d) : M is the set, d is the *metric.*
  - $d : M \times M \to \mathbb{R}$ must satisfy the following properties $\forall x, y, z \in M$,
    - $d(x, y) = 0 \iff x = y$ (Distance to itself is 0)
    - $d(x, y) = d(y, x)$ (Symmetry)
    - $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)

- Euclidean metric



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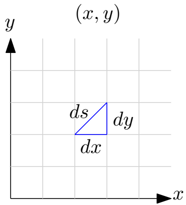
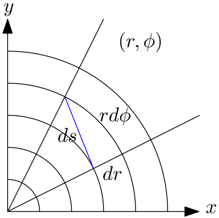
Familiar examples

- Line element : Use the same metric, to get an equality for infinitesimal distances (Again, Physicist's viewpoint)
- Line element in *Rectangular/Cartesian* and *Polar* coordinates





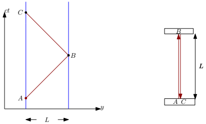
$$ds^2 = dx^2 + dy^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

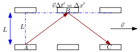
$$ds^2 = dr^2 + rd\phi^2 = \begin{bmatrix} dr & d\phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} dr \\ d\phi \end{bmatrix}$$

- Our goal : ***Build a Line element in SR, i.e. to find an object that let's us measure the infinitesimal distance between two points on our 3+1 "Spacetime".***

Gedanken experiment : Light clock (Stationary)

- Stationary Light clock





- Light ray travels a distance L twice and it travels at c in the ($A \to B \to C$ path)

$$c = \frac{2L}{\Delta t} \quad \left(\text{Speed} = \frac{\text{Distance}}{\text{Time}}\right)$$

$$\Rightarrow \Delta t = \frac{2L}{c}$$

- For the events A and C in the *unprimed* frame

$$\Delta x = \Delta y = \Delta z = 0$$

$$\Delta t = \frac{2L}{c}$$

Gedanken experiment : Light clock (Moving)

- Now, let us make the clock move along the x-axis, with speed $\vec{v} = (v_x, 0, 0)$
  - In this situation, the photons have to travel a little more in the $A \to B \to C$ path
  - Yet, from basic symmetry, we can see that $d(A, B) = d(B, C) = L'$

$$L' = \sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$

$$\text{where} \Delta x' = v_x \Delta t'$$

  - Giving us

$$2L' = 2\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$

$$(\text{Total distance travelled } = 2L')$$

  - $\Delta t'$ is the time from $A \to C$

$$\Delta t' = \frac{2L'}{c} = \frac{2}{c}\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$

Gedanken experiment : Combine results and define an invarient

- In Stationary

$$\begin{aligned}-(c\Delta t)^2+(\Delta\vec{r})^2 &= -(c\Delta t)^2+(\Delta x)^2+(\Delta y)^2+(\Delta z)^2\\ &= -(c\Delta t)^2\\ &= -(2L)^2 = -4L^2\end{aligned}$$

- When moving

$$\begin{aligned}-(c\Delta t')^2+(\Delta\vec{r}')^2 &= -4\left(L^2+\left(\frac{\Delta x'}{2}\right)^2\right)+(\Delta\vec{r}')^2\\ &= -4\left(L^2+\left(\frac{\Delta x'}{2}\right)^2\right)+(\Delta x')^2+(\Delta y')^2+(\Delta z')^2\\ &= -4L^2-(\Delta x')^2+(\Delta x')^2+0+0\\ &= -4L^2\end{aligned}$$

- Giving us

$$\begin{aligned}-(c\Delta t)^2+(\Delta\vec{r})^2 &= -(c\Delta t')^2+(\Delta\vec{r}')^2\\ (\Delta s')^2 &= (\Delta s)^2\end{aligned}$$

Infinitesimal limit

- From $\Delta \to d$

$$\begin{aligned} ds^2 &= -c\,dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c\,dt^2 + d\vec{r}^2 \\ &= \begin{bmatrix} c\,dt & dx & dy & dz \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\eta_{\mu\nu}} \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix} \\ &= \sum_{\mu} \sum_{\nu} \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \end{aligned}$$

- Where

$$x^{\mu} = (ct, x, y, z) = (ct, \vec{r})$$

- Notation

$$\left[x^{\mu} = \left(x^0, x^1, x^2, x^3\right) = \left(x^0, \vec{x}\right)\right] = \left[r^{\mu} = \left(r^0, r^1, r^2, r^3\right) = \left(r^0, \vec{r}\right)\right]$$

Galilean transformation vs Lorentz Transformation

- Galilean Transformation

$$
\begin{aligned}
t' &= t \\
x' &= 1\,(x - vt) \\
y' &= y \\
z' &= z
\end{aligned}
$$

- Galilean and Lorentz transformation describes transformation of coordinates of a point K in K' (both inertial frames) - like rotation in 3D (*Use Dustin*)
  - All physical quantities that transform under rotation like $\vec{r}$ does $\to$ Vectors
    - Similarly, think how scalars are defined
  - All physical quantities that transform under Lorentz boosts like x^μ $\to$ 4-vectors
- Special Lorentz Transformation in x-direction

$$
\begin{aligned}
t' &= \gamma\left(t - \frac{\beta x}{c}\right) \to t' = \gamma\,(t - \beta x) \\
x' &= \gamma\,(x - \beta t) \\
y' &= y \\
z' &= z
\end{aligned}
$$

  - where

$$\vec{\beta} = \frac{\vec{v}}{c} = \beta \vec{e}_{\mathbf{1}}, \quad \gamma = \frac{1}{\sqrt{1-\beta^{2}}} \text{ Lorentz factor}$$

- To go to inverse Lorentz transformation

$$x'^{\mu} \leftrightarrow x^{\mu}, \beta \leftrightarrow -\beta$$

4-vector definition

- Matrix form / 4-vector form

$$\underbrace{\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}}_{x'^{\mu}} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Lambda^{\mu}_{\nu}} \underbrace{\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}}_{x^{\nu}}$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \left(= \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu} \right)$$

- Arbitrary 4-vector : $A^{\mu} = (A_0, A_1, A_2, A_3) = \left(A_0, \vec{A}\right)$

$$A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu}$$

Lorentz Transformation in arbitrary direction

- In an arbitrary direction $\vec{\beta} = \left(\frac{\vec{v}}{c}\right) = \left(\frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right) \rightarrow \vec{\beta} = (v_x, v_y, v_z)$
  - Split $\vec{r}$

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$$
$$\vec{r}_{\parallel} = \frac{\vec{r} \cdot \vec{\beta}}{\beta} \frac{\vec{\beta}}{\beta}$$
$$\vec{r}_{\perp} = \vec{r} - \frac{\vec{r} \cdot \vec{\beta}}{\beta} \frac{\vec{\beta}}{\beta}$$

  - Lorentz transformation in direction β

$$r^{\mathbf{0}\,\prime} = \gamma \left(r^{\mathbf{0}} - \vec{\beta} \cdot \vec{r}\right)$$
$$\vec{r}_{\parallel}' = \gamma \left(\vec{r}_{\parallel} - \beta x^{\mathbf{0}}\right)$$
$$\vec{r}_{\perp}' = \vec{r}_{\perp}$$

4-vectors (Summary)
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- Arbitrary 4-vector→ $A^\mu = \left(A_0, \vec{A}\right)$
  - Transformation laws

$$A^{0\prime} = \gamma \left(A^0 - \vec{\beta}\cdot\vec{A}\right)$$
$$A'_\parallel = \gamma \left(A_\parallel - \beta A_0\right)$$
$$\vec{A}'_\perp = \vec{A}_\perp$$

  - Invariance

$$-\left(A'_0\right)^2 + \left|\vec{A}'\right|^2 = -\left(A_0\right)^2 + \left|\vec{A}\right|^2$$

  - Two 4 vectors form an "invariant" scalar product

$$A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu = \begin{bmatrix} A^0 & \vec{A} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B^0 \\ \vec{B} \end{bmatrix}$$

- Some 4-vector examples
  - (Arbitrary) $A^\mu = \left(A_0, \vec{A}\right)$
  - (4-Position) $x^\mu = \left(x^0, \vec{x}\right) = \left(ct, \vec{x}\right)$
  - (4-Velocity) - Next time

https://en.wikipedia.org/wiki/Euclidean_distance