Special Relativity

A crash course of the main results that one should have seen before doing GR

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Contents

Contents

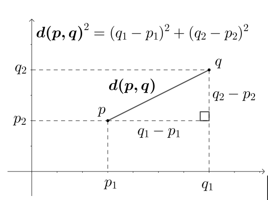
1 Minkowski Metric

Metric

- What is a **metric**? What is it's purpose? (Also called as the *line element* - especially in Physics literature)
  - The term metric comes from "metric space". Metric space is a very basic structure that we can put on a set.

Metric space

- Metric space : set → together with a *metric* on the set.
  - The metric → function → defines a concept of distance → between any two members of the set, which are usually called points.
- Formal definition : A metric set is an ordered pair (M, d) : M is the set, d is the *metric.*
  - $d : M \times M \to \mathbb{R}$ must satisfy the following properties $\forall x, y, z \in M$,
    - $d(x, y) = 0 \iff x = y$ (Distance to itself is 0)
    - $d(x, y) = d(y, x)$ (Symmetry)
    - $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)
- Euclidean metric



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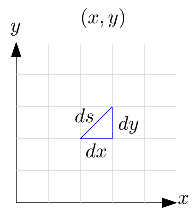

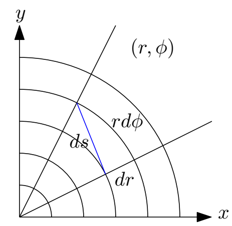

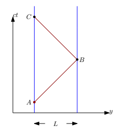

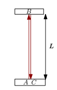

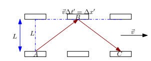

Familiar examples

- Line element : Use the same metric, to get an equality for infinitesimal distances (Again, Physicist's viewpoint)
- Line element in *Rectangular/Cartesian* and *Polar* coordinates

$$ds^2 = dx^2 + dy^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$ds^2 = dr^2 + rd\phi^2 = \begin{bmatrix} dr & d\phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} dr \\ d\phi \end{bmatrix}$$

- Our goal : ***Build a Line element in SR, i.e. to find an object that let's us measure the infinitesimal distance between two points on our 3+1 "Spacetime".***

Gedanken experiment : Light clock (Stationary)

- Stationary Light clock
- Light ray travels a distance L twice and it travels at c in the ($A \to B \to C$ path)

$$c = \frac{2L}{\Delta t} \quad \left(\text{Speed} = \frac{\text{Distance}}{\text{Time}}\right)$$
$$\Rightarrow \Delta t = \frac{2L}{c}$$

- For the events A and C in the *unprimed* frame

$$\Delta x = \Delta y = \Delta z = 0$$
$$\Delta t = \frac{2L}{c}$$

Gedanken experiment : Light clock (Moving)

- Now, let us make the clock move along the x-axis, with speed $\vec{v} = (v_x, 0, 0)$
  - In this situation, the photons have to travel a little more in the $A \to B \to C$ path
  - Yet, from basic symmetry, we can see that $d(A, B) = d(B, C) = L'$

$$L' = \sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$
$$\text{where} \Delta x' = v_x \Delta t'$$

– Giving us

$$2L' = 2\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$

(Total distance travelled $=2L'$)

– $\Delta t'$ is the time from $A \to C$

$$\Delta t' = \frac{2L'}{c} = \frac{2}{c}\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}$$

Gedanken experiment : Combine results and define an invarient

- In Stationary

$$\begin{aligned}
-(c\Delta t)^2 + (\Delta \vec{r})^2 &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\
&= -(c\Delta t)^2 \\
&= -(2L)^2 = -4L^2
\end{aligned}$$

- When moving

$$\begin{aligned}
-(c\Delta t')^2 + (\Delta \vec{r}')^2 &= -4\left(L^2 + \left(\frac{\Delta x'}{2}\right)^2\right) + (\Delta \vec{r}')^2 \\
&= -4\left(L^2 + \left(\frac{\Delta x'}{2}\right)^2\right) + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \\
&= -4L^2 - (\Delta x')^2 + (\Delta x')^2 + 0 + 0 \\
&= -4L^2
\end{aligned}$$

- Giving us

$$\begin{aligned}
-(c\Delta t)^2 + (\Delta \vec{r})^2 &= -(c\Delta t')^2 + (\Delta \vec{r}')^2 \\
(\Delta s')^2 &= (\Delta s)^2
\end{aligned}$$

Infinitesimal limit

- From $\Delta \to d$

$$\begin{aligned}
ds^2 &= -c\,dt^2 + dx^2 + dy^2 + dz^2 \\
&= -c\,dt^2 + d\vec{r}^2 \\
&= \begin{bmatrix} c\,dt & dx & dy & dz \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\eta_{\mu\nu}} \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix} \\
&= \sum_{\mu}\sum_{\nu} \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu}
\end{aligned}$$

- Where

$$x^{\mu} = (ct, x, y, z) = (ct, \vec{r})$$

- Notation

$$\left[x^{\mu} = \left(x^0, x^1, x^2, x^3\right) = \left(x^0, \vec{x}\right)\right] = \left[r^{\mu} = \left(r^0, r^1, r^2, r^3\right) = \left(r^0, \vec{r}\right)\right]$$

2 Recall : Lorentz Transformation and it's properties

Galilean transformation vs Lorentz Transformation

- Galilean Transformation

$$\begin{aligned} t' &= t \\ x' &= 1\,(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

- Galilean and Lorentz transformation describes transformation of coordinates of a point K in K' (both inertial frames) - like rotation in 3D (*Use Dustin*)
  - All physical quantities that transform under rotation like $\vec{r}$ does $\rightarrow$ Vectors
    - Similarly, think how scalars are defined
  - All physical quantities that transform under Lorentz boosts like x^μ $\rightarrow$ 4-vectors
- Special Lorentz Transformation in x-direction

$$\begin{aligned} t' &= \gamma\left(t - \tfrac{\beta x}{c}\right) \rightarrow t' = \gamma\,(t - \beta x) \\ x' &= \gamma\,(x - \beta t) \\ y' &= y \\ z' &= z \end{aligned}$$

  - where

$$\vec{\beta} = \frac{\vec{v}}{c} = \beta \vec{e}_1, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \text{ Lorentz factor}$$

- To go to inverse Lorentz transformation

$$x'^\mu \leftrightarrow x^\mu, \beta \leftrightarrow -\beta$$

4-vector definition

- Matrix form / 4-vector form

$$\underbrace{\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}}_{x'^\mu} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Lambda^\mu_\nu} \underbrace{\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}}_{x^\nu}$$

$$x'^\mu = \Lambda^\mu_\nu x^\nu \left(= \sum_\nu \Lambda^\mu_\nu x^\nu \right)$$

- Arbitrary 4-vector : $A^\mu = (A_0, A_1, A_2, A_3) = \left(A_0, \vec{A}\right)$

$$A'^\mu = \Lambda^\mu_\nu A^\nu$$

Lorentz Transformation in arbitrary direction

- In an arbitrary direction $\vec{\beta} = \left(\frac{\vec{v}}{c}\right) = \left(\frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right) \rightarrow \vec{\beta} = (v_x, v_y, v_z)$
  - Split $\vec{r}$

$$\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$$

$$\vec{r}_\parallel = \frac{\vec{r}\cdot\vec{\beta}}{\beta}\,\frac{\vec{\beta}}{\beta}$$

$$\vec{r}_\perp = \vec{r} - \frac{\vec{r}\cdot\vec{\beta}}{\beta}\frac{\vec{\beta}}{\beta}$$

- Lorentz transformation in direction β

$$r^{0\prime} = \gamma\left(r^0 - \vec{\beta}\cdot\vec{r}\right)$$
$$\vec{r}_{\parallel}' = \gamma\left(\vec{r}_{\parallel} - \beta x^0\right)$$
$$\vec{r}_{\perp}' = \vec{r}_{\perp}$$

4-vectors (Summary)
/

- Arbitrary 4-vector→ $A^\mu = \left(A_0, \vec{A}\right)$
  - Transformation laws

$$A^{0\prime} = \gamma\left(A^0 - \vec{\beta}\cdot\vec{A}\right)$$
$$A_{\parallel}' = \gamma\left(A_{\parallel} - \beta A_0\right)$$
$$\vec{A}_{\perp}' = \vec{A}_{\perp}$$

  - Invariance

$$-\left(A_0'\right)^2 + \left|\vec{A}'\right|^2 = -\left(A_0\right)^2 + \left|\vec{A}\right|^2$$

  - Two 4 vectors form an "invariant" scalar product

$$A^\mu B_\mu = \eta_{\mu\nu}A^\mu B^\nu = \begin{bmatrix} A^0 & \vec{A} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B^0 \\ \vec{B} \end{bmatrix}$$

- Some 4-vector examples
  - (Arbitrary) $A^\mu = \left(A_0, \vec{A}\right)$
  - (4-Position) $x^\mu = \left(x^0, \vec{x}\right) = (ct, \vec{x})$
  - (4-Velocity) - Next time

References

[1] https://en.wikipedia.org/wiki/Euclidean_distance