GR in a nutshell

A quick summary of the framework

Rohan Kulkarni

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Contents

Introductory textbooks

- Relativity, Gravitation and Cosmology by *Lambourne*
  - Super easy to read. Easiest text I've come across for GR
- A first course in General relativity by *Bernard Schutz**
  - Used by Prof. Amendola. Standard text used for an advanced undergrad course in GR
- General relativity - An introduction for Physicists by *Hobson***
  - One of my favorites. Clean, upto the point and gets the job done precisely
- Gravity by *Hartle**
  - Another one of my favorites. A lot of text to read but worth it. More time consuming than Hobson in my opinion.
- Einstein's General relativity by *D'Inverno* *

Advanced textbooks

There are many more advanced books out there. Interested people can have a chat with me after the course is done.

- General relativity by *Wald****
  - My first exposure to the subject (Wouldn't wish that even on my worst enemy)
    - - Why use it as a first book then? My BSc thesis was based on topics from Part II of the book.
  - But, the holy grail/bible of modern GR. Everything you need to know about modern GR is there.
  - Do not pick it up if you have no previous exposure to the subject.
- Gravitation by *Misner, Throne, Wheeler* **
  - Written by three experts who are responsible for a ridiculous amount of development in the subject.
  - The telephone book of GR. Over 1200 pages.
  - Do not try to read it linearly at-least as of now. It's a reference book when you really want to get an intuition on the topic.

One equation to rule them all

- GR - *"Relativistic field theory of Gravity"*
- "*Spacetime tells matter how to move; matter tells spacetime how to curve*" - *John Wheeler*

$$\underbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}}_{\text{Telling matter how to move}} = \underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{\text{Telling spacetime how to curve}}$$

- Everything on the LHS has to do with $g_{\mu\nu}$ (metric tensor).
  - You use $g_{\mu\nu}$ to compute $R_{\mu\nu\lambda\kappa}$ (Riemann tensor). Using $R_{\mu\nu\lambda\kappa}$ one can easily (with tedious algebra) compute $R_{\mu\nu}$(Ricci tensor) and R (Ricci scalar).
- Everything on the RHS has to do with $T_{\mu\nu}$

GR vs SR

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

General relativity

- $g_{\mu\nu}$ is anything that solves the EE for a particular $T_{\mu\nu}$
- $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$

Special relativity

- $g_{\mu\nu} = \eta_{\mu\nu} = (-1, +1, +1, +1)$ for $T_{\mu\nu} = 0$
- $\frac{d^2 x^\mu}{d\tau^2} + \cancel{\Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau}} = 0$, as $\Gamma^\mu{}_{\nu\sigma} = 0$
- Mainly because $\frac{dx^\mu}{d\tau} = c$

Conclusion

Special relativity is just really a "special" case of GR. $(\nabla_\mu \to \partial_\mu, g_{\mu\nu} \to \eta_{\mu\nu})$

A common tale of two equations describing classical theories

- Main two objects of **classical** field theories
  - Fields
  - Particles (*Trajectory*)
- Electromagnetism
  - The **fields** themselves are governed by ***Maxwell's equations*** (Can get $\vec{E}, \vec{B}$ for given $A^{\mu} = \left(\phi, \vec{A}\right)$ or for given $J^{\mu} = \left(\rho, \vec{J}\right)$)
  - The **motion of a test particle** is dictated by ***Lorentz force*** :$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$
- GR (Just like above, we will have two equations)
  - The **fields** themselves are governed by ***Einstein's equations*** (Calculate $g_{\mu\nu}$ for the given $T^{\mu\nu}$)
  - The **motion of a test particle** are governed by the ***geodesic equation*** $\left(\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = 0\right)$: Dot is differentiation with respect to proper time
  - The RHS of this equation is *zero* for *free-fall objects in gravity*.
    - What would happen if we have free *fall gravity + electrodynamics force*?
    - $\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = \frac{q}{m_0} F^{\mu}_{\nu}\dot{x}^{\nu}$

A Geodesic equation you have seen before

Recall

- **Principle of least action** $\rightarrow S\left[x^i(t)\right] = \int_{t_1}^{t_2} dt\, L\left(x^i(t), \dot{x}^i(t)\right)$ (Remember : A functional)
- If you see how the action changes for a *small perturbation* in the path $\rightarrow$ $x^i(t) \rightarrow x^i(t) + \delta x^i(t)$
  Keep the end points fixed $\rightarrow \delta x^i(t_1) = \delta x^i(t_2) = 0$
- Now if you compute $\delta S \rightarrow$ You will get the ***Euler Lagrange*** *equations*

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

- If we use this in *flat space for a particle* with just kinetic energy i.e. $\rightarrow$ $L = \frac{1}{2}m\left(\dot{x^i}\right)^2$
- We get
  - (Newton's law) $m\ddot{x}^i = 0$
  - Which makes sense because $\rightarrow F = m\ddot{x}^i = \left(-\frac{\partial V}{\partial x^i} = -\nabla V\right)$
- You could solve the Newton's law for a particle to get its *trajectory* (this is all happening in Flat Euclidean 3 space $\left(\mathbb{R}^3\right)$)
- Our immediate goal is to get the *write down the Lagrangian and action* $\rightarrow$ **for particles in curved space**
  (Eventually spacetime)

Non-relativistic motion of a particle in curved space

- Lagrangian of a free particle in flat space

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

  - Also can be rewritten using

$$(g_{\textbf{flat}})_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \frac{1}{2} m \ (g_{\textbf{flat}})_{ij} \, \dot{x}^i \dot{x}^j$$

  - Euler Lagrange equation gives us $\ddot{x}^i = 0$
- For any arbitrary $g_{ij}(\vec{x})$,

$$L = \frac{1}{2} m \, g_{ij}(\vec{x}) \, \dot{x}^i \dot{x}^j$$

  - We will use Euler-Lagrange equations on it → **Geodesic equation**

Calculating the geodesic equation for non-relativistic particle in curved space

$$L = \frac{1}{2} m \, g_{ij} \left(\vec{x}\right) \dot{x}^i \dot{x}^j$$

- Recall E-L equation

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

  - LHS

$$\frac{\partial L}{\partial x^i} = \frac{m}{2} \frac{\partial g_{jk}}{\partial x^i} \, \dot{x}^j \dot{x}^k$$

  - RHS

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(m g_{ik}\left(x^j\right) \dot{x}^k\right) = m \frac{\partial g_{ik}}{\partial x^j} \dot{x}^j \dot{x}^k + m g_{ik} \ddot{x}^k$$

  - LHS=RHS

$$g_{ik} \ddot{x}^k + \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i}\right) \dot{x}^j \dot{x}^k = 0$$

$$g_{ik} \ddot{x}^k + \frac{1}{2}\left(2 \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) \dot{x}^j \dot{x}^k = 0$$

$$g_{ik} \ddot{x}^k + \frac{1}{2}\left(\underbrace{\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k}}_{\textbf{Symmetry with } j \leftrightarrows k} - \frac{\partial g_{jk}}{\partial x^i}\right) \dot{x}^j \dot{x}^k = 0$$

The geodesic equation in curved space

- Inverse metric

$$\underbrace{g^{ij} g_{jk}}_{\text{Matrix multiplication}} = \delta^i_k$$

- Recall the equation we derived in the slide before

$$g_{ik}\ddot{x}^k + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k = 0$$

  - now,

$$g_{ik}\ddot{x}^k + \underbrace{\frac{1}{2}g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)}_{\Gamma^i_{jk}=\Gamma^i_{kj}}\dot{x}^j\dot{x}^k = 0$$

  - giving us

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

- This is known as the **geodesic equation** and its solutions are known as ***geodesics.***