GR in a nutshell

A quick summary of the framework

Rohan Kulkarni

April 16, 2021

Contents

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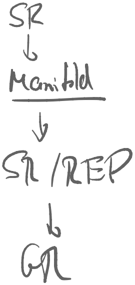

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- Einstein's General relativity by *D'Inverno* * K-calculus / Chp. 6.

David Tong's notes.

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Singularity thms,
BH mechanics,
Global structure
IVP,

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One equation to rule them all

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Metric tensor   Matter

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{Telling matter how to move}} = \underbrace{\frac{8\pi G}{c^4} T_{\mu\nu}}_{\text{Telling spacetime how to curve}}$$

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GR vs SR

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \kappa T_{\mu\nu}$$

General relativity

Special relativity

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Conclusion

Special relativity is just really a "special" case of GR. $(\nabla_\mu \to \partial_\mu, g_{\mu\nu} \to \eta_{\mu\nu})$

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A common tale of two equations describing classical theories

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ds, dy, dx | $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ Minkowski space $\rightarrow$ $x^\mu = (t, x, y, z) = (t, \vec{r})$

Pseudo Riemannian

$\mathbb{R}^4 \longrightarrow x^\mu = (x, y, z, w)$ $\quad ds^2 = dx^2 + dy^2 + dz^2 + dw^2$

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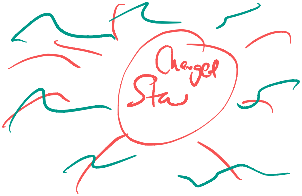

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$F = m\ddot{x} = -kx$

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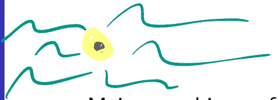

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$\ddot{x}^\mu = 0$ (SR) No grav

$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0$ (GR)

A Geodesic equation you have seen before

Recall

- **Principle of least action** $\rightarrow S\left[x^i\left(t\right)\right] = \int_{t_1}^{t_2} dt\; L\left(x^i\left(t\right), \dot{x}^i\left(t\right)\right)$ (Remember : A functional)

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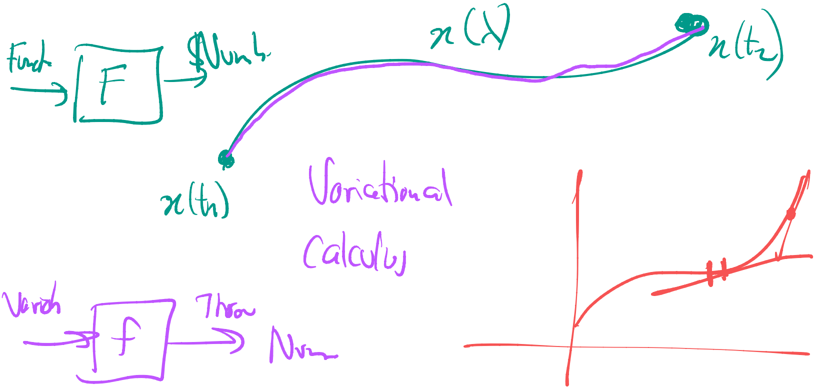

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- Now if you compute $\delta S \rightarrow$ You will get the ***Euler Lagrange*** *equations*

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

1D : $L = \frac{1}{2} m\dot{x}^2$

Free particle.

$\dot{x} = \frac{dx}{dt}$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$0 = \frac{d}{dt}\left(\frac{2}{2} m\dot{x}\right) = m\ddot{x}$$

$m\ddot{x} = 0 \rightarrow$ Geodesic eqn for particle in flat space with no force acting on it.

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- If we use this in *flat space for a particle* with just kinetic energy i.e. $\rightarrow$ $L = \frac{1}{2} m \left(\dot{x}^i\right)^2$

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A Geodesic equation you have seen before

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$i = 1, 2, 3$

$m\ddot{x}^1 = 0 \Rightarrow m\ddot{x} = 0$

$m\ddot{x}^2 = 0 \Rightarrow m\ddot{y} = 0$

$m\ddot{x}^3 = 0 \Rightarrow m\ddot{z} = 0$

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- Our immediate goal is to get the *write down the Lagrangian and action* $\rightarrow$ **for particles in curved space**
  (Eventually spacetime)

Non-relativistic motion of a particle in curved space

- Lagrangian of a free particle in flat space (Repeat of past slide)

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

Non-relativistic motion of a particle in curved space

$\vec{A} = (A_x, A_y, A_z) = (A_1, A_2, A_3)$
$= A_i$

- Lagrangian of a free particle in flat space

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  - Also can be rewritten using

$$(g_{\textbf{flat}})_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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  - We will use Euler-Lagrange equations on it $\rightarrow$ **Geodesic equation**

Calculating the geodesic equation for non-relativistic particle in curved space

$$L = \frac{1}{2} m \, g_{ij} \left(\vec{x} \right) \dot{x}^i \dot{x}^j$$

- Recall E-L equation

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$$g_{ik}\ddot{x}^k + \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\right)\dot{x}^j \dot{x}^k = 0$$

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The geodesic equation in curved space

- Inverse metric

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- This is known as the **geodesic equation** and its solutions are known as ***geodesics.***