Causal Structure in General Relativity

Hauptseminar on Quantum field theory and Gravity, SoSe 2018

Rohan Kulkarni
Institute of Theoretical Physics, Leipzig (Germany)

This is the transcript of the talk that was given at ITP,Leipzig for the Hauptseminar on quantum field theory and gravity in Summer Semester 2018 (SoSe 2018). The main goal of the talk was to introduce the notions of causal structure needed to build up singularity theorems and many more modern general relativity topics.

Keywords
General relativity, Causal structure, Causality conditions, Domains of dependence, Cauchy surfaces

The main topics covered in this talk were as follows:

- Definitions of causal relations between events.
- Different orders of causality on an arbitrary spacetime.
  - Simply causal spacetime
  - Strongly causal spacetime.
  - Stably causal spacetime.
- Domains of dependence.
- Cauchy surfaces.

The literature used for this talk were the following books :

1. General Relativity , Robert.M. Wald.
2. The large scale structure of spacetime, Hawking and Ellis.

Causal Structure

We know *solution to Einstein's equation* is some *Metric tensor* g_{ab} which is associated with a *4-Dimensional Manifold* (Spacetime). Let us from now on denote spacetime with (M, g). This metric g on any given point $p \in M$ defines a scalar product between any two vectors in the *tangent space* T_pM. This also defines the norm of any single vector in T_pM. According to the sign of the norm we have classified the vectors as

1. $g_{ab}v^a v^b < 0$ →Timelike
2. $g_{ab}v^a v^b = 0$ →Null
3. $g_{ab}v^a v^b > 0$ →Spacelike

Remark. Time orientability

At every event $p \in M$ the *tangent space* is *isomorphic* to the *minkowski spacetime.* We have light cones through every event in this manifold. A light cone passing through the origin of T_pM is *defined* as the **light cone of** p.Light cone of p is a subset of T_pM and not of M .

So, for each of this point p we have a light cone and each of this light cone has a "*future*" and a "*past*". If a *continuous* designation of *future* and *past* can be made over the whole manifold M then we call our spacetime (M, g) is **Time orientable.** The spacetimes that we have dealt with till now and the ones that we will deal with will all be time orientable.

Since a tangent vector v_0 at a given event $p_0 \in M$ can be thought as a *velocity* vector of a test particle passing through a point $p_0 = \gamma(t_0)$ at a given instant of time t_0. The curve or trajectory $\gamma(t)$ along which this point takes a path can be identified as *spacelike, timelike or null. Timelike* and *Null* can be further segregated as future as past directed (We will define this in the upcoming segments).

1 Definitions/Theorems of Future and Pasts

Definition 1. Character of a point at a curve

A ***curve*** $\gamma(t)$ is **timelike (past or future directed)** , **null (past or future directed)** or **spacelike** at $p_0 = \gamma(t)$ if its ***tangent vector*** $\dot{}\,(t_0)$ at p_0 is *timelike (past or future directed), null(past or future directed) or spacelike.*

Definition 2. Global character of curve

- A *curve* γ which is *timelike* at every event is a ***timelike curve***
- A *curve* γ which is *spacelike* at every event is a ***spacelike curve.***
- A *curve* γ which is *null* at every event is a ***null curve.***

Event: Point on our manifold M

Definition 3. Future and Past directed curve.

- A *timelike or a null* ***curve*** which lies in the ***future half*** of the *light cone* is called a **Future directed curve.**
- A *timelike or a null* ***curve*** which lies in the ***past half*** of the *light cone* is called a **Past directed curve.**

2 Definitions/Theorems of Causal relations between events

Definition 4. Chronological Future and Chronological Past of an event $p \in M \rightarrow (I^{+}(p)$ and $I^{-}(p))$

For a specific given event $p \in M$, the ***chronological future*** is defined by the set of all points that can be reached by a *future directed timelike curve* which starts at p. This set of points is denoted by $I^{+}(p)$.

For a specific given event $p \in M$, the ***chronological past*** is defined by the set of all points that can be reached by a *past directed timelike curve* which starts at p. This set of points is denoted by $I^{-}(p)$.

Remark. For a *set* of points $\mathcal{U}$, the chronological future/past is defined as follows:

$$I^{\pm}(\mathcal{U}) = \bigcup_{p \in \mathcal{U}} I^{\pm}(p)$$

$I^{\pm}(\mathcal{U})$ is the "Chronological Future/Past" of $\mathcal{U}$.

Definition 5. Causal Future and Past of an event $p \in M \rightarrow (J^{+}(p)$ and $J^{-}(p))$

For a given event $p \in M$, the ***causal future*** is defiend by the set of all points that can be reached by a *future directed "null" or a "timelike" curve* which starts at p. This set of points is denoted by $J^{+}(p)$.

For a given event $p \in M$, the ***causal past*** is defined by the set of all points that can be reached by a *past directed "nul l" or "timelike" curve* which starts at p. This set of points is denoted by $J^{-}(p)$

Remark. For a *set* of points $\mathcal{U}$, the causal future/past is defined as follows:

$$J^{\pm}(\mathcal{U}) = \bigcup_{p \in \mathcal{U}} J^{\pm}(p)$$

$J^{\pm}(\mathcal{U})$ is the "Causal Future/Past" of $\mathcal{U}$.

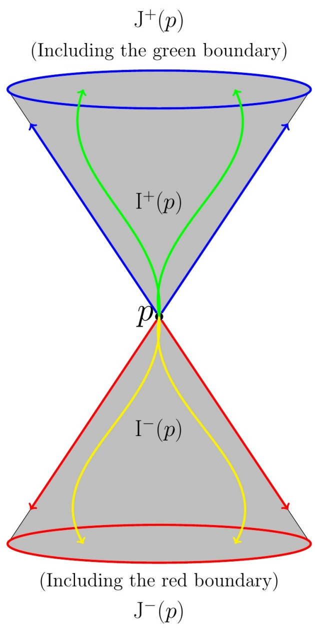


Figure 1: Example of a Lightcone

Definition 6. Future endpoint of a curve.

Let λ be a future directed causal curve in M. We say that $p \in M$ is a ***future endpoint*** of λ if for every neighbourhood O or p $\exists t_0$ such that $\lambda(t) \in O$ for all $t > t_0$.
Thus by *hausdorff property of* M we can have at most one future endpoint

Definition 7. Future inextendible curve.

A causal curve γ_c is called ***future inextendible*** if it does not have a **future endpoint.**

Definition 8. Past endpoint of a curve.

Let λ be a past directed causal curve in M. We say that $p \in M$ is a ***past endpoint*** of λ if for every neighbourhood O or p $\exists t_0$ such that $\lambda(t) \in O$ for all $t < t_0$.
Thus by *hausdorff property of* M we can have at most one past endpoint.

Definition 9. A causal curve γ_c is called ***past inextendible*** it does not have a **past endpoint.**

Definition 10. Achronal Sets

Achronal sets $\mathcal{A}$ are subsets of spacetime M that hold the property $\mathcal{A} \cap I^+(\mathcal{A}) = \emptyset$.

Remark. Intuitively what this means is that in these sets there are no such events which are in the future of another event in the set. Imagine a set of events S (points in a spacetime) and any future event of these points does not belong to the set S. Such a set S is called an ***Achronal set.***

No two events in an *Achronal set* are *Causally connected* to each other. Now what do I mean by causally connected? What I mean is , "No two events in an *Achronal set* can be connected to each other by *null or timelike* curves."

Definition 11. Let $\mathcal{A}$ be an *Achronal set.* Let $\partial\mathcal{A}$ be the *Edge* of the *Achronal set.* Then $\partial\mathcal{A}$ is the subsets of all the events $p \in \mathcal{A}$ such that every neighbourhood of p, $\mathcal{U} \in M$ contains at least a point $p_+ \in I^+(p)$ and $p_- \in I^-(p)$ and a *timelike curve* γ_T connecting $p_+$and p_- where $\gamma_T \cap \mathcal{A} = \emptyset$

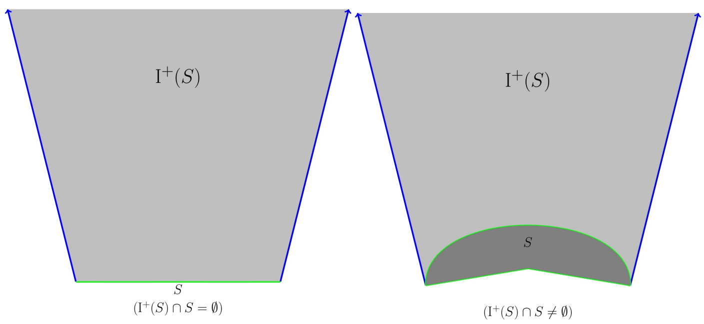


Figure 2: Example of an Achronal Set

3 Definitions of Causality , Initial Conditions

We say that our Spacetime (M, g) is ***causal*** if it does not contain a *closed causal (timelike or null) curve.*

This definition of a spacetime being **causal** has a few drawbacks. Particularly speaking, if our spacetime is arbitrarily close to being causal it could allow *timelike curves* which are not "closed" but arbitrarily close to being "*closed*". We don't like this because this allows us intuitively to "nearly" go back in time.

As an example lets look at this diagram where the cylinder is our spacetime and the lightcones on it define our causality.

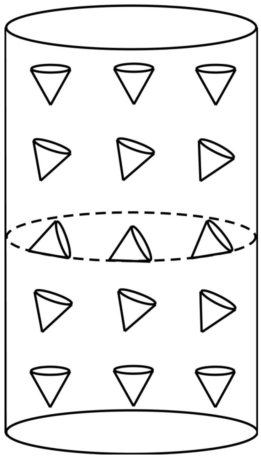


Figure 3: Cylindrical spacetime

In this spacetime as you can see if the cones become even a little bit more horizontal then intuitively we will be having timelike curves which are tending to become closed timelike curves. Such a spacetime is causal but barely causal i.e. ***"not Strongly causal".*** So let us now define what does it mean by a Strongly Causal spacetime.

Definition 12. Strongly Causal
(M,g) is called **Strongly causal** if :

$\forall p\,,\, p \in M$ and $\forall$ neighborhoods $\mathcal{U}$ of p there is a neighborhood $V \subseteq \mathcal{U}$ such that :

- No *causal* curve γ intersects V more than once.
  - Indeed, If (M, g) is not *Strongly causal* $\Rightarrow$There exists a causal curve γ which comes arbitrarily close to intersecting itself.
  - We require strong causality to keep causal curves at least a finite distance from intersecting themselves.

Remark. Spacetimes in which the events from the future can influence their past i.e. spacetimes in which there are closed causal curves **do not** satisfy **Strong causality.** From this point of view strong causality seems like a sensible physical requirement.

Now it seems that the definition of *Strong causality* should be enough to do physics on our spacetime as it gives us an sensible condition that events from the future cannot effect events in their past. But there is still a small issue. We have not done anything to avoid this condition of non-existence of closed timelike curves in case of small perturbations. Lets deal with that now and we will have a causal structure of a spacetime with which we can work. For that we will define ***Stably Causal.*** But before denying that lets consider our problem with ***Strong causality*** and then define ***stable causality*** as a result of that problem

Problem. Arbitrarily small perturbations in the metric somewhere could allow causal curves to self intersect. Find a condition to avoid this. (We will define this condition as the necessary condition for Stable causality

Solution. Let us setup a few things first:

- Consider a perturbing metric g through :

$$g_{mn} \to \tilde{g_{mn}} = g_{mn} - \omega_m \omega_n \tag{1}$$

with a *timelike cotangent vector field* ω_{mn}(AIN→ Abstract Index notation).
  - So we have two metrics on the same differentiable manifold, g_{mn} and $\tilde{g}_{mn}$. g_{mn} is Lorrentzian / Pseudo Riemannian.

- Notice : $\tilde{g_{mn}}$ still has the same signature but "Light cones are now wider for $\tilde{g_{mn}}$."
  - How to see this?
    * Compare $g_{mn}v^m v^n$ and $\tilde{g_{mn}}v^m v^n$. (Calculate the length of square of tangent vector v from the tangent vector field v^a at point p with respect to the two metrics)

$$\tilde{g_{mn}} v^m v^n = g_{mn} v^m v^n - v^m \omega_m v^n \omega_n \tag{2}$$

Now look at the second term on the right hand side carefully if $v^m w_m = v^n \omega_n = \alpha \in \mathbb{R}$

$$v^m \omega_n v^m \omega_n = \alpha \cdot \alpha = \alpha^2 > 0 \tag{3}$$

      · This means that in 2 the right hand side is smaller than the left hand side without the second term

$$\tilde{g_{mn}} < g_{mn}$$

Thus, it is easier for vectors v to have $||v|| < 0$ i.e. to be timelike or null for $\tilde{g_{mn}}$ than for g_{mn} because of 3.

- So in conclusion what we say is that , "Some vector that is *spacelike* with respect to g_{mn} ***maybe*** *timelike with respect to* $\tilde{g_{mn}}$.
- $(M, \tilde{g})$ spacetime has all *causal curves* of (M, g) plus more curves.
  - i.e. {Causal curves of (M, g)} $\subseteq$ {Causal curves of $(M, \tilde{g})$}

So we have solved our problem by perturbing our metric and still maintaining the condition for causality. This condition where the perturbed metric also stays sensibly causal is know as **Stably Causal.**

Definition 13. Stably Causal
(M, g) is called *Stably causal* if there exists a covector field ω_a such that $(M, \tilde{g})$ is also causal. ($\tilde{g}_{\mu\nu} = g_{\mu\nu} - \omega_\mu \omega_\nu$)

Some quick theorems on Stable Causality without proofs. (Proofs can be found in Wald's Book)

Theorem. *If*(M, g) *is* ***stably causal*** *then it* ***implies*** *that* (M, g) *is* ***strongly causal.***

Theorem. (M, g) *is stably causal iff there exists a differentiable function* $f \in C^\infty(M, \mathbb{R})$ *such that* $\nabla^a f$ *is a* ***past directed timelike vector field*** *or* $-\nabla^a f$ *is a* ***future directed timelike vector field.***

Remark. Intuitively, this means that f can be viewed as a **cosmic clock**. (Not a unique one as we can have more than one f satisfying our conditions)

Recall : We defined something called "*Time orientability of a spacetime*" at the start of this section. We said that :
(M, g) is *time orientable* iff there exists a *past/future pointing smooth timelike vector field.* What separates this from the theorem above the remark is that here the *smooth timelike vector field* need not be a *Gradient field.*

4 Definitions/Theorems of Global Dependence and Cauchy Surfaces.

This section is the final buildup of the *causal structure* that we would eventually need to effectively define a ***Black hole*** in this causal structure language***.*** (We still will need some definitions from Asymptotic Flatness which we will define further)

Definition 14. Future domain dependence of set S. ($D^+(S)$)

Assume $S \subseteq M$ is a *closed achronal set.* Then the **future domain of dependence of S** is defined as

$$D^+(S) \quad := \quad \{p \in M \,|\, \text{Every past inextendible causal curve through } p \text{ intersects } S\ \}$$

Definition 15. Past domain of dependence of set S. ($D^-(S)$)

$$D^-(S) \quad := \quad \{p \in M \,|\, \text{Every future inextendible causal curve through } p \text{ intersects } S\ \}$$

Remark. The set of events p that affect only S.

Definition 16. Domain of Dependence. ($D(S)$)

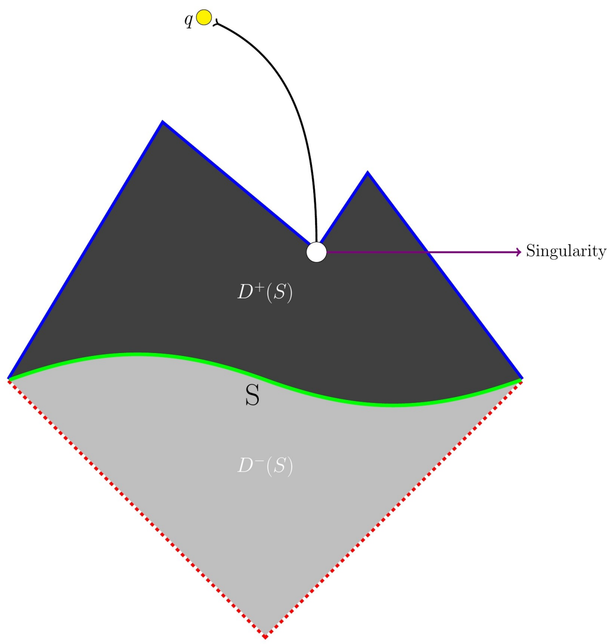
This is just defined as the union of the *future and past domain of dependence.*

$$D(S) = D^+(S) \cup D^-(S)$$

Remark. The total domain of dependence of set S *i.e.* both past and future events affected by S.

Example. Example for future domain of dependence.

Figure 4: Domain of Dependence



Looking at the figure above, One question we want to ask is, "Why $q \notin D^+(M)$?"

The answer is quite simple. For q , some *past inextendible curve* does not intersect S because it gets stuck at the *singularity.* In other words we can say, "q is affected by events in the *shadow* of the singularity".

Definition 17. Future Cauchy horizon of S. ($H^+(S)$)

$$H^+(S): \quad = \quad \overline{D^+(S)} - I^-\left(D^+(S)\right)$$

$\overline{D^+(S)}$: Closed set of future domain of dependence of S.
$I^-(D^+(S))$: Chronological past of the future domain of dependence of S.

Remark. $H^+(S)$ is ***achronal.*** This is quite obvious as no two events in $H^+(S)$ are *causally connected* to each other.

This is the set of latest events that are effected only by S.

Definition 18. Past Cauchy Horizon : $H^-(S)$

$$H^-(S): \quad = \quad \overline{D^-(S)} - \overline{I^+(D^-(S))}$$

Remark. Set of earliest events that affect only S.

Definition 19. *Full Cauchy* ***horizon*** *of S.* (H(S))

$$H(S) \quad = \quad H^+(S) \cup H^-(S)$$

Proposition. *Full Cauchy horizon is just the boundary of Full domain of dependence.*

$$H(S) \quad = \quad \dot{D}(S)$$

Now lets define probably the most important definition of the whole chapter. We were building up all the definitions to get to this definition.

Definition 20. ***Cauchy Surface.***

A *closed, achronal set S* is called a "**Cauchy Surface**", if its *full cauchy horizon vanishes.* That is, if

1. $H(S) = \emptyset$
2. $\dot{D}(S) = \emptyset$
3. $D(S) = M$

Look at point **3** carefully, the domain of dependence of S is the entire manifold M (Spacetime). What this physically means is that, if you know what happened on S (Initial conditions with suitable evolution laws) then you can predict what can happened on the entire manifold M (i.e. the entire spacetime).

- This is the reason why ***Cauchy surfaces*** are so important. If the conditions on a *Cauchy surface* are known then everything on M can be predicted.
- Since a *Cauchy surface* is *Achronal* , it can be viewed as an instant in time. (As no two events are connected causally on an *Achronal set*)
- The term *Surface* in "Cauchy Surface is motivated by the following theorem:
  - Theorem : *Every Cauchy surface* Σ *, is a 3D sub-manifold* (C^0) *of M.*

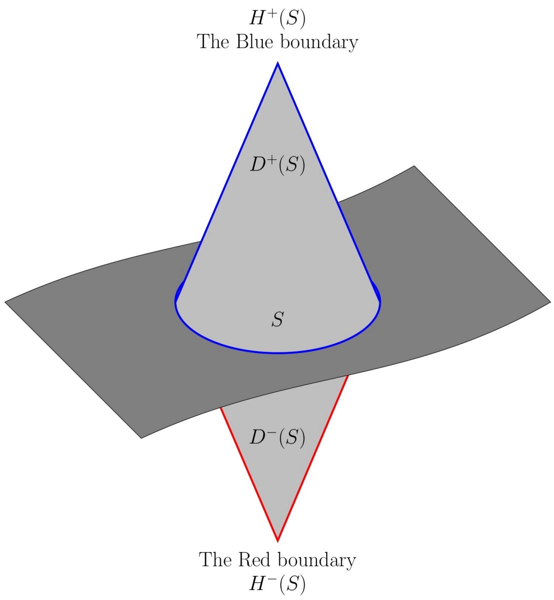


Figure 5: Cauchy Surface