Problem. Vector current for antifermions

In this exercise you will show that the vector current of a Dirac fermion with four-momentum $p = (p_0, \vec{p})$ is directly related to the vector current of its antifermion with four-momentum $p' = (p_0, -\vec{p})$.

In the course of the exercise you will encounter the charge conjugate Dirac spinor, defined as $\psi_c \equiv C\overline{\psi}^\top$, where C is the charge conjugation operator. It is convenient to work in the Dirac representation of the Clifford algebra, where $C = i\gamma_2\gamma_0$.

a) Show that the charge conjugation operator fulfills the following relations

$$-C = C^T = C^\dagger = C^{-1} \,. \tag{2}$$

Here C^{-1} is the inverse of C.

b) Using Eq. (2), show that the vector current of a Dirac field, $j^\mu = \overline{\psi}\gamma^\mu\psi$, is equal to the vector current of the charge conjugate field, $j^\mu_c = \overline{\psi}_c\gamma^\mu\psi_c$, *i.e.*, that

$$j^\mu = j^\mu_c \,. \tag{3}$$

c) For the spinors in momentum space, the charge conjugation properties of the fermion field imply that

$$u_s(p) = C\overline{v}_s^\top(p) \,. \tag{4}$$

Use this relation to show the the vector current of a fermion with momentum p is connected to the vector current of its antifermion with momentum p' via

$$\overline{u}_s(p)\gamma_\mu u_s(p) = \overline{v}_s(p')\gamma^\mu v_s(p') \,. \tag{5}$$

Hint: A relation between spinors with opposite momenta was derived in the lecture.

Problem 2. Vector current for antifermions

In this exercise you will show that the vector current of a Dirac fermion with four-momentum $p = (p_0, \vec{p})$ is directly related to the vector current of its antifermion with four-momentum $p' = (p_0, -\vec{p})$. In the course of the exercise you will encounter the charge conjugate Dirac spinor, defined as $\psi_c \equiv C\overline{\psi}^\top$, where C is the charge conjugation operator. It is convenient to work in the Dirac representation of the Clifford algebra, where $C = i\gamma_2\gamma_0$.

a) Show that the charge conjugation operator fulfills the following relations

$$-C = C^T = C^\dagger = C^{-1}\,. \tag{2}$$

Here C^{-1} is the inverse of C.

→ Since $C = i\gamma_2\gamma_0$ we have :

$$-C = -i\gamma_2\gamma_0 = i\gamma_0\gamma_2 \Rightarrow -CC = i\gamma_0\gamma_2 i\gamma_2\gamma_0 = -\gamma_0\overbrace{\gamma_2\gamma_2}^{-\mathbb{1}}\gamma_0 = (\gamma_0)^2 = \mathbb{1} \Rightarrow \boxed{C^{-1} = C}$$

In the Dirac representation it becomes clear, that $C = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ so that $\boxed{-C = C^T = C^\dagger}$

b) Using Eq. (2), show that the vector current of a Dirac field, $j^\mu = \overline{\psi}\gamma^\mu\psi$, is equal to the vector current of the charge conjugate field, $j^\mu_c = \overline{\psi}_c\gamma^\mu\psi_c$, *i.e.*, that

$$j^\mu = j^\mu_c\,. \tag{3}$$

$(\gamma^2)^\dagger = -\gamma^2$

We know $\psi \xrightarrow{C} (-i)\gamma^2\psi^\dagger(x)$; $\overline{\psi} \xrightarrow{C} (-i\gamma^2\psi^\dagger(x))^\dagger\gamma^0 = i\psi(x)^T(-\gamma^2)\gamma^0$

$$\Rightarrow \overline{\psi}_c\gamma^\mu\psi_c = \psi(x)^T(-\gamma^2)\gamma^0\gamma^\mu\gamma^2\psi^T(x) = \overline{\psi}(x)\gamma^0(-\gamma^2\gamma^0\gamma^\mu\gamma^2)^T\psi(x) = \overline{\psi(x)}\gamma^0(\gamma^0\gamma^2\gamma^\mu\gamma^2)^T\psi(x)$$

$$= \overline{\Psi(x)}\gamma^0\left(\gamma^0\gamma^2\left(-\gamma^2\gamma^\mu + 2\eta^{\mu 2}\mathbb{1}\right)\right)^T\Psi(x) = \overline{\Psi(x)}\gamma^0\left(\gamma^0\gamma^\mu + 2\eta^{\mu 2}\gamma^0\gamma^2\right)^T\Psi(x)$$

The term in the transposed bracket is $(\gamma^0\gamma^\mu)^T$ for $\mu \neq 2$ & $-\gamma^0\gamma^\mu$ for $\mu = 2$, So that we can write this (γ^2 is purely complex entries)

$$j_c^\mu = \overline{\Psi_c}\gamma^\mu\Psi_c = \overline{\Psi}(x)\gamma^0(\gamma^0\gamma^\mu)^T\Psi(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x) = j^\mu$$

c) For the spinors in momentum space, the charge conjugation properties of the fermion field imply that

$$u_s(p) = C\overline{v}_s^\top(p)\,. \tag{4}$$

Use this relation to show the the vector current of a fermion with momentum p is connected to the vector current of its antifermion with momentum p' via

$$\overline{u}_s(p)\gamma_\mu u_s(p) = \overline{v}_s(p')\gamma^\mu v_s(p')\,. \tag{5}$$

Hint: A relation between spinors with opposite momenta was derived in the lecture.

Using $u_s(p) = C\,\overline{v_s(p)}^T = C\gamma^0 v_s(p)^\dagger$ we find $\overline{u_s(p)} = v_s(p)^T(\gamma^0)^\dagger C^\dagger\gamma^0$

$$= -v_s(p)^T\gamma^0 C\gamma^0 = v_s(p)^T C$$

As a consequence of this: $\overline{u_s(p)}\gamma_\mu u_s(p) = v_s(p)^T C\gamma_\mu C\,\overline{v_s(p)}^T = \overline{v_s(p)}\,(C\gamma_\mu C)^T v_s(p)$

Now we use: $v_s(p) = -\gamma^0 v_s(p')$, $\overline{v_s(p)} = \overline{v_s(p')}(-\gamma^0)$ so that:

$$\overline{u_s(p)}\gamma_\mu u_s(p) = \overline{v_s(p')}\gamma^0 C\gamma_\mu^T C\gamma^0 v_s(p') = \overline{v_s(p')}\gamma^2\gamma_\mu^\dagger\gamma^2 v_s(p') = \eta_{\mu\nu}\overline{v_s(p')}\gamma^2(\gamma^\nu)^T\gamma^2 v_s(p')$$

$$= \eta_{\mu\nu}\overline{v_s(p')}\left(\gamma^2\gamma^\nu\gamma^2\right)^T v_s(p') = \eta_{\mu\nu}\overline{v_s(p')}\left(\gamma^\nu + 2\eta^{\nu 2}\gamma^2\right)^T v_s(p')$$

$$= \eta^{\mu\nu}\,\overline{v_s(p')}\,\eta_{\nu\alpha}\gamma^{\alpha}v_s(p') = \delta^{\mu}_{\alpha}\,\overline{v_s(p')}\,\gamma^{\alpha}v_s(p') = \overline{v_s(p')}\,\gamma^{\mu}v_s(p')$$