Parity transformations of Fermion bilinears

Problem statement

We have

$$\psi(t,\vec{x}) \xrightarrow{\mathcal{P}} \eta\gamma^0\psi(t,-\vec{x})$$

where $\eta \in \mathbb{C}$ and $|\eta| = 1$

(a) Derive the transformation properties of the following fermion bilinears under parity:

  (a) Pseudo-scalar current: $\bar{\psi}i\gamma_5\psi$

  (b) Axial-vector current: $\bar{\psi}\gamma^\mu\gamma_5\psi$

  Which structures are invariant under a Parity transformation, and which are not? Check it explicitly.

  Hints: First of all, check that $\bar{\psi}(t,\vec{x}) \xrightarrow{\mathcal{P}} \eta^*\bar{\psi}(t,-\vec{x})\gamma^0$. Then, you should make use of the properties of the γ matrices as representations of the Clifford algebra, in particular, $\left(\gamma^0\right)^2 = 1$; $\left\{\gamma^0,\gamma^k\right\} = 0$

(b) Show explicitly that electromagnetic fermion interactions are invariant under CP transformations.

  Hint: Remember that under charge conjugation: $C\bar{\psi}\gamma^\mu\psi C^{-1} = -\bar{\psi}\gamma^\mu\psi$ and $CA_\mu C^{-1} = -A_\mu$

(c) Show that the helicity of Dirac fermion changes sign under space reflection but not under time reversal.

  Hint: remember the helicity operator is defined as $\hat{\Lambda} = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ with $\overrightarrow{\hat{\Sigma}} = \frac{1}{2}\begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}$ the spin operator. To check the behavior under time reversal, adopt explicit expressions for the $u(p)$ spinor and restrict to the case $p_x = 0 = p_y$.

[P1.] Parity transformations of Fermion bilinears.

$$\Psi(t,\vec{x}) \xrightarrow{P} \eta\gamma^0\Psi(t,-\vec{x}) \qquad \begin{matrix}\eta\in\mathbb{C}\\ |\eta|=1\end{matrix}$$

(a) Derive the transformation properties for the following fermion bilinears under pairity

→ Pseudo-scalar current : $\overline{\Psi}i\gamma_5\Psi$

→ Axial-vector current : $\overline{\Psi}\gamma^\mu\gamma_5\Psi$

Which structures are invariant under a Pairity transfo, which aren't?

→ Let's check the hint:

$$\overline{\Psi}(t,\vec{x}) = \Psi^\dagger(t,\vec{x})\gamma^0 \xrightarrow{P} \left(\eta\gamma^0\Psi(t,\vec{x})\right)^\dagger\gamma^0$$

$$= \eta^*\Psi^\dagger(t,-\vec{x})\gamma^0\gamma^0$$

$$= \eta^*\overline{\Psi}(t,-\vec{x})\gamma^0$$

→ $$\overline{\Psi}(t,\vec{x})\,i\gamma_5\,\underbrace{\Psi(t,\vec{x})} \xrightarrow{P} \eta^*\overline{\Psi}(t,-\vec{x})\,i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\,\underbrace{\eta\gamma^0\Psi(t,-\vec{x})}$$

$$= |\eta|^2\,\overline{\Psi}(t,-\vec{x})\,i\,(-1)\gamma^5\Psi(t,-\vec{x})$$

$$= \cancel{|\eta|^2\,\overline{\Psi}(t,-\vec{x})\,i\,(-1)}$$

$$= -\overline{\Psi}(t,-\vec{x})\,i\gamma^5\Psi(t,-\vec{x})$$

↳ Gets an additional '-' sign ⇒ Not invariant.

→ $$\overline{\Psi}\gamma^\mu\gamma_5\Psi \xrightarrow{P} \overline{\Psi(t,-\vec{x})}\gamma^0\gamma^\mu\gamma_5\gamma^0\Psi(t,-\vec{x}) = \overline{\Psi(t,-\vec{x})}\left(-(\gamma^\mu\gamma_5)^\dagger\right)\Psi(t,-\vec{x})$$

$$= \begin{cases} -\overline{\Psi(t,-\vec{x})}\gamma^0\gamma_5\Psi(t,\vec{x}) & \mu=0\\ \Psi(t,\vec{x})\gamma^i\gamma_5\Psi(t,-\vec{x}) & \mu=i\in\{1,2,3\} \end{cases}$$

→ Spatial comp are invariant.

b) Show explicitly that electromagnetic fermion interactions are invariant under CP transformations.

Hint: remember from the lecture that under charge conjugation: $C\bar{\psi}\gamma^\mu\psi C^{-1} = -\bar{\psi}\gamma^\mu\psi$ and $CA_\mu C^{-1} = -A_\mu$.

The interaction term is given by (for QED) : $\bar{\Psi}\cancel{A}\Psi$

$$\bar{\Psi}\cancel{A}\Psi \xrightarrow{P} \bar{\Psi}(t,-x)\gamma^0 A_0(t,-\vec{x})\Psi(t,-\vec{x}) - \bar{\Psi}(t,-x)\gamma^i(-A_i(t,-\vec{x}))\Psi(t,-\vec{x})$$

$$\xrightarrow{C} -\Psi(t,-\vec{x})\gamma^0\Psi(t,-\vec{x})(-A_0(t,-\vec{x})) + \bar{\Psi}(t,-\vec{x})\gamma^i\Psi(t,-\vec{x})A_i(t,\vec{x})$$

$$= \bar{\Psi}(t,-\vec{x})\gamma^\mu A_\mu\Psi(t,-\vec{x})$$

Where we used : $\bar{\Psi}\gamma^\mu\Psi \xrightarrow{P} \begin{cases} \bar{\Psi}\gamma^\mu\Psi(t,-\vec{x}) & \text{for } \mu=0 \\ -\Psi\gamma^\mu\Psi(t,-\vec{x}) & \text{else} \end{cases}$

$$\& \quad A^\mu(t,\vec{x}) \xrightarrow{P} \begin{cases} A^\mu(t,-\vec{x}) & \text{for } \mu=0 \\ A^\mu(t,-\vec{x}) & \text{for } \mu=i\in\{1,2,3\} \end{cases}$$

c) Show that the helicity of a dirac fermion changes sign under space reflection, but not under time reversal.

Hint: remember the helicity operator is defined as $\hat{\Lambda} = \frac{\hat{\vec{\Sigma}}\cdot\vec{p}}{|\vec{p}|}$ with $\hat{\vec{\Sigma}} = \frac{1}{2}\begin{pmatrix}\sigma^i & \\ & \sigma^i\end{pmatrix}$ the spin operator. To check the behavior under time reversal, adopt explicit expressions for the $u(p)$ spinor and restrict to the case $p_x = 0 = p_y$.

- We know $\vec{p} \xrightarrow{\mathcal{P}} -\vec{p}$ but $\vec{\hat{\Sigma}} \xrightarrow{\mathcal{P}} \vec{\hat{\Sigma}}$ $\Rightarrow \hat{\Lambda} = \frac{\vec{\hat{\Sigma}}\cdot\vec{p}}{|\vec{p}|} \xrightarrow{\mathcal{P}} \frac{\vec{\hat{\Sigma}}\cdot(-\vec{p})}{|\vec{p}|} = -\hat{\Lambda}$
- Similarly $\vec{p} \xrightarrow{\mathcal{T}} -\vec{p}$ and $\vec{\hat{\Sigma}} \xrightarrow{\mathcal{T}} -\vec{\Sigma}$ $\Rightarrow \hat{\Lambda} \xrightarrow{\mathcal{T}} \hat{\Lambda}$
- Now, to see the transformation change of $\hat{\Sigma}$ under $\underline{\mathcal{P}}$ & $\underline{\mathcal{T}}$ resp. Consider e.g. $\Sigma^{(3)}$

$$\Sigma^{(3)} = \left(\frac{1}{2}\frac{u_{s=\frac{1}{2}}(p)\,\overline{u_{s=\frac{1}{2}}(p)}}{2m} - \frac{1}{2}\frac{u_{s=-\frac{1}{2}}(p)\,\overline{u_{s=-\frac{1}{2}}(p)}}{2m}\right)\delta(p-n) \qquad \text{——} \circledast$$

- We can consider $\vec{p} = \vec{e}_n$ then $u_s(p) = \begin{pmatrix}\sqrt{p_0+m}\;\chi_s \\ \sqrt{p_0-m}\;\sigma_3\chi_s\end{pmatrix}$

and $u_s(p) \xrightarrow{\mathcal{P}} u_s(p)$; $\overline{u_s(p)} \xrightarrow{\mathcal{P}} \overline{u_s(p)}$

- Since $p' \xrightarrow{\mathcal{P}} p'$, as a consequence $\hat{\Sigma}^{(3)} \xmapsto{\mathcal{P}} \hat{\Sigma}$ because of $\circledast$
- In a similar manner because $p_0 \xrightarrow{\mathcal{T}} -p_0$

We find $u_s(p) \xrightarrow{\mathcal{T}} \begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix} u_s(p)$; $\overline{u_s(p)} \xrightarrow{\mathcal{T}} u_s(p)\begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix}$ & in combo with

$$\rightarrow \hat{\Sigma}^{(3)} \xrightarrow{\mathcal{T}} \begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix}\hat{\Sigma}^{(3)}\begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix} = \begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}0 & i\sigma^3 \\ i\sigma^3 & 0\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = -\hat{\Sigma}^{(3)}$$

Analogously we can show $\hat{\vec{\Sigma}} \xrightarrow{T} -\hat{\vec{\Sigma}}$