Problem. $e^-\mu^-$ Scattering

Compute the S matrix element for genuine $e^-(p_a) + \mu^-(p_b) \to e^-(p_1) + \mu^-(p_2)$ scattering in QED, i.e. neglect the case when no scattering takes place, $p_a = p_1$.

a) Write down the contributing Feynman graph(s) to $|\mathcal{M}|^2$ in the leading order approximation.

b) Apply the QED Feynman rules (cf. the script for Lecture 5) and translate the Feynman diagrams into an algebraic expression for the amplitude $i\mathcal{M}$.

c) Compute the unpolarized matrix element $|\mathcal{M}|^2$, i.e. sum over final state spins, and average over initial state spins. Neglect the electron and muon masses for convenience. You will need the trace identity

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) , \tag{1}$$

and the rule for complex conjugation of the spinor product

$$[\bar{u}(p)\gamma^\mu u(k)]^\star = \bar{u}(k)\gamma^\mu u(p) \tag{2}$$

(no proofs needed). Express the unpolarized $|\mathcal{M}|^2$ in terms of the Mandelstam variables.

d) What happens for forward scattering

$$\theta_{a1} \to 0, \qquad \text{i.e. } \cos\theta_{a1} = \frac{\vec{p}_a \cdot \vec{p}_1}{|\vec{p}_a|\,|\vec{p}_1|} \to 1\,? \tag{3}$$

Do you find this result surprising? Which of the assumptions we made might be responsible for this result?

Rohan Kulkarni

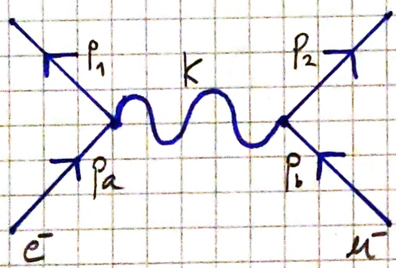
P1. $e^-\mu^-$ Scattering.

- Compute the S-matrix element for genuine $e^-(p_a) + \mu^-(p_b) \to e^-(p_1) + \mu^-(p_2)$

Scattering in QED i.e neglect the case when no scattering takes place, $p_a = p_1$

(a) & (b)

- Write down the contributing Feynman diagrams to $|M|^2$ in the leading order approximation
- Apply QED Feynman rules & translate the Feynman diagram into an algebraic expression for $i\mathcal{M}$

$i\mathcal{M} =$ 

$$= (-ie)\bar{u}(p_1)\gamma^\mu u(p_a) \frac{-i\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{(p_1^2 - p_a^2)} (-ie)\bar{u}(p_2)\gamma^\nu u(p_b)$$

$$k^\mu = p_a^\mu - p_1^\mu$$

Extra

The $k^\mu k^\nu$ term drops out for on-shell spinors (Also expected by gauge invariance)

↓ Check / Verify

$$\bar{u}_\alpha(p_1)\gamma^\mu_{\alpha\beta} u_\beta(p_a) k^\mu = \bar{u}_\alpha(p_1)\gamma^\mu_{\alpha\beta} u_\beta(p_a) p_a^\mu + \bar{u}_\alpha(p_1)\gamma^\mu_{\alpha\beta} u_\beta(p_a) p_1^\mu$$

$$= \bar{u}_\alpha(p_1)\cancel{p}_a u_\beta(p_a) + \bar{u}_\alpha(p_1)\cancel{p}_1 u_\beta(p_a)$$

On shell spinors

$$= m\,\bar{u}_\alpha(p_1)u_\beta(p_a) - m\,\bar{u}_\alpha(p_1)u_\beta(p_a)$$

$$= 0$$

$$\mathcal{M} = \frac{e^2}{t}\,\bar{u}(p_1)\gamma^{\mu}u(p_a)\cdot\bar{u}(p_2)\gamma_{\mu}u(p_b)$$

$$t = (p_1 - p_3)^2$$

(c) Compute the unpolarized matrix element $|\mathcal{M}|^2$, i.e. sum over final state spins, and avg over initial state spins. Neglect the electron & muon masses for convinience.

You will need

$$\rightarrow \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

$$\rightarrow [\bar{u}(p)\gamma^{\mu}u(k)]^{*} = \bar{u}(k)\gamma^{\mu}u(p)$$

Express the unpolarized $|\mathcal{M}|^2$ in terms of Mandelstam variables.

Sol^n: $$\mathcal{M}^{\dagger} = \frac{e^2}{t}\,\bar{u}(p_b)\gamma^{\mu}u(p_2)\cdot\bar{u}(p_a)\gamma_{\mu}u(p_1)$$

Giving us:

‘Mu’ ‘Nu’

$$|\mathcal{M}|^2 = \frac{e^4}{t^2}[\bar{u}(p_1)\gamma^{\mu}u(p_a)][\bar{u}(p_2)\gamma_{\mu}u(p_b)][\bar{u}(p_b)\gamma^{\nu}u(p_2)][\bar{u}(p_a)\gamma_{\nu}u(p_1)]$$

(This is a contraction of two tensors, one depending only on initial state & other only on final state)

~~Now we will go to the part where we sum over final spin sums.~~

$$\cancel{= \frac{e^4}{t^2}[\bar{u}(p_1)\gamma_{\mu}u(p_a)][\bar{u}(p_a)\gamma}$$

Using the argument from the bracket we can ~~reag~~ rearrange the following way:

$$|M|^2 = \frac{e^4}{t^2} \underbrace{[\bar{u}(p_1)\gamma^\mu u(p_a)][\bar{u}(p_a)\gamma^\nu u(p_1)]}_{\circledast} \cdot \underbrace{[\bar{u}(p_2)\gamma_\mu u(p_b)][\bar{u}(p_a)\gamma_\nu u(p_1)]}_{\tilde{\circledast}}$$

• Now we will go to the next part where we sum over the final states

$$\circledast \rightarrow \sum_s \sum_{s'} [\bar{u}^{s'}(p_1)\gamma^\mu u^s(p_a)][\bar{u}^{s}(p_a)\gamma^\nu u^{s'}(p_1)]$$

$$= \sum_s \sum_{s'} [\bar{u}^{s'}_\beta(p_1)\gamma^\mu_{\beta\delta} (u^s(p_a)\bar{u}^{s}(p_a))_{\delta\lambda} \gamma^\nu_{\lambda\alpha} u^{s'}_\alpha(p_1)]$$

$$= \sum_{s'} [\bar{u}^{s'}_\beta(p_1)\gamma^\mu_{\beta\delta} (\not{p}_a + m_e \mathbb{1})_{\delta\lambda} \gamma^\nu_{\lambda\alpha} u^{s'}_\alpha(p_1)]$$

$$= [(\not{p}_1 + m_e\mathbb{1})_{\alpha\beta} \gamma^\mu_{\beta\delta} (\not{p}_a + m_e\mathbb{1})_{\delta\lambda} \gamma^\nu_{\lambda\alpha}]$$

$$= Tr[(\not{p}_1 + m_e)\gamma^\mu(\not{p}_a + m_e)\gamma^\nu]$$

Similarly

$$\tilde{\circledast} \rightarrow \sum_s \sum_{s'} [\bar{u}^{s}(p_2)\gamma_\mu u^{s'}(p_b)][u^{s'}(p_b)\gamma_\nu \bar{u}^{s}(p_2)]$$

$$\vdots$$

$$= Tr[(\not{p}_2 + m_{\mu^-})\gamma_\mu(\not{p}_b + m_{\mu^-})\gamma_\nu]$$

Both these traces can be evaluated using γ-matrix identities.

• Let us assume we don't know the polarization of the initial states (For multiple measurements we would then assume that we get an avg of the states)

$$\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \quad \left(\tfrac{1}{2} \text{ for each incoming } e^- \,\&\, \mu^-\right)$$

$$\rightarrow \frac{1}{4}\sum_{spin}|M|^2 = \frac{e^4}{4t^2}\, Tr\left[(\not p_1 + m_e)\gamma^\mu(\not p_a + m_e)\gamma^\nu\right]$$
$$Tr\left[(\not p_2 + m_\mu)\gamma_\mu(\not p_b + m_\mu)\gamma_\nu\right]$$

- Now use the trace identities:

$\cancel{Tr[(\not p_2 + m_\mu)\gamma_\mu}$

$$\cancel{Tr\left[(\not p_1 + m_e)\gamma^\mu(\not p_1 + m_e)\gamma^\nu\right] = p_1^\rho p_a^\sigma\, Tr[\gamma_\rho \gamma}$$

~~These have γ-matrices in ...~~

$$Tr\left[(\not p_1 + m_e)\gamma^\mu(\not p_a + m_e)\gamma^\nu\right] = (p_1^\rho)(p_a^\sigma)\, Tr\left[\gamma_\rho\gamma^\mu\gamma_\sigma\gamma^\nu\right]$$
$$- m_e^2\, Tr\left[\gamma^\mu\gamma^\nu\right]$$

$$= 4\left(p_1^\mu p_a^\nu + p_a^\mu p_1^\nu - (p_1)^\rho (p_a)_\rho\, \eta^{\mu\nu}\right) - 4m_e^2\, \eta^{\mu\nu}$$

- Similarly can compute for the Tr with m_μ.

$$\rightarrow \frac{1}{4}\sum_{spin}|M|^2 = \frac{4e^4}{t^2}\left[p_a^\mu p_1^\nu + p_1^\mu p_a^\nu - \left((p_a)^\rho(p_1)_\rho + m_e^2\right)\eta^{\mu\nu}\right]$$
$$\left[p_b^\mu p_2^\nu + p_2^\mu p_b^\nu - \left((p_2)^\sigma(p_b)_\sigma + m_\mu^2\right)\eta^{\mu\nu}\right]$$

Using new notation

$$\cancel{= \frac{4e^4}{t^2}\cdot 2\left(p_{ab}p_{12} + p_{a2}p_{1b} + p_{1b}p_{a2} + \right.}$$

$$= \frac{4e^4}{t^2}\cdot 2\left[p_{ab}p_{12} + p_{a2}p_{1b} + m_\mu^2 p_{a1} + m_e^2 p_{b2} + 2m_e^2 m_\mu^2\right]$$

$$\Big\downarrow \text{ Neglecting masses}$$

$$= \frac{8e^4}{t^2}\left[p_{ab}p_{12} + p_{a2}p_{1b}\right]$$

Using Mandelstam variable.

$$\rightarrow s = (p_a + p_b)^2 = p_a^2 + p_b^2 + 2p_{ab} \approx m_e^2 + m_\mu^2 + 2p_{ab}$$

$$= 2p_{ab} \Rightarrow \boxed{p_{ab} = \frac{s}{2}}$$

$$\rightarrow u = (p_b - p_1)^2 = \ldots = -2p_{b1}$$

$$\Rightarrow \boxed{p_{b1} = \frac{-u}{2}} = p_{a2}$$

Giving us

$$\Rightarrow |M|^2 = \frac{2e^4}{t^2}(s^2+u^2)$$

(d) What happens to forward scattering $\theta_{a1} \to 0$ i.e

$$\cos\theta_{a1} = \frac{\vec{p}_a \cdot \vec{p}_1}{|\vec{p}_a||\vec{p}_1|} \to 1?$$

Do you find this result surprising? Which of the assumptions we made might be resp. for this result?

Sol. d) From mandelstam relations:

$$t = -s\,\frac{1-\cos\theta^*}{2}$$

$$u = -s\,\frac{1+\cos\theta^*}{2}$$

• From (c):

$$\frac{s^2+u^2}{t^2} = \frac{s^2 + s^2\left(\frac{1+\cos\theta^*}{2}\right)^2}{s^2\left(\frac{1-\cos\theta^*}{2}\right)^2} = \frac{\cancel{s^2} + \cancel{s^2}\left(\frac{1}{4} + \frac{\cos\theta^*}{2} + \frac{\cos^2\theta^*}{4}\right)}{\cancel{s^2}\left(\frac{1}{4} - \frac{\cos\theta^*}{2} + \frac{\cos^2\theta^*}{4}\right)}$$

$$= \frac{1+\cos^4\left(\frac{\theta^*}{4}\right)}{\sin^4\left(\frac{\theta^*}{4}\right)}$$

for $\theta^* \to 0$: $\dfrac{1+\cos^4\left(\frac{\theta^*}{2}\right)}{\sin^4\left(\frac{\theta^*}{2}\right)} \to \infty \Rightarrow \dfrac{s^2+u^2}{t^2} \to \infty$