Problem. Bhabha Scattering

In this problem we will show that the differential cross section for Bhabha scattering ($e^+e^- \to e^+e^-$) is given by

$$\frac{d\sigma}{d\text{cos}\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] \tag{1}$$

using a result from last week's tutorial

$$d\sigma = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 d\Omega \tag{2}$$

a) Write down the two Feynman diagrams that contribute to this process at the first order in α. Why is there a relative minus sign between the two diagrams?

b) Use the QED Feynman rules to show that the amplitude can be written as

$$i\mathcal{M} = \frac{ie^2}{t}\mathcal{A}_1 - \frac{ie^2}{s}\mathcal{A}_2\,, \tag{3}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are products of spinors and gamma matrices.

c) Calculate $|\mathcal{M}|^2$, in the high energy limit (neglect the electron mass)

(1) Use trace identities to show that the first term is given by

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{A}_1|^2 = 2(u^2+s^2) \tag{4}$$

Note that this is the same expression obtained for the $e\mu$ scattering process in problem sheet 2, in the massless limit.

(2) Calculate (or use crossing symmetry) the second term, $\frac{1}{4}\sum_{\text{spins}}|\mathcal{A}_2|^2$.

(3) Use the identities $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$ and $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4(a\cdot b)$ to show that the two interference terms are given by

$$\frac{1}{4}\sum_{\text{spins}}\mathcal{A}_1\mathcal{A}_2^* = \frac{1}{4}\sum_{\text{spins}}\mathcal{A}_2\mathcal{A}_1^* = -2u^2. \tag{5}$$

d) Add up the different contributions and use Eq. (2) to translate the matrix element into the differential cross section $d\sigma/d\cos\theta$.

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P.1. Bhatha Scattering

In this problem we will show that the differential cross section for Bhatha scattering ($e^+e^- \to e^+e^-$) is given by

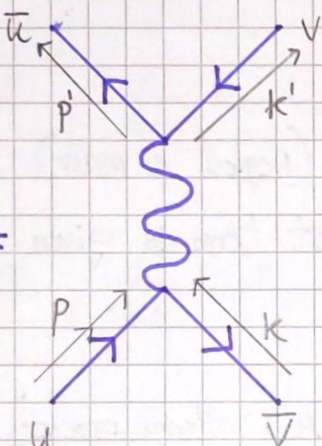
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]$$

Using a result from last week's tut

$$d\sigma = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 d\Omega$$

(a) Write down the two Feynman diagrams that contribute to the process at first order in α. Why is there a relative minus sign between the two diagrams?

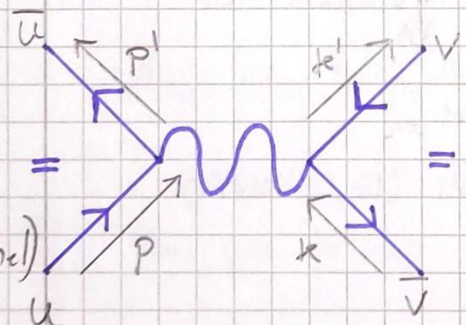
Sol^n:

$i\mathcal{M}_s$ = 

$= \frac{ie^2}{s}\eta_{\alpha\beta}\,\bar{v}(k)\gamma^\alpha u(p)\,\bar{u}(p')\gamma^\beta v(k')$

(s-channel)

\+ \+ \+

$i\mathcal{M}_t$ = 

$= \frac{ie^2}{t}\eta_{\mu\nu}\,\bar{u}(p')\gamma^\mu u(p)\,\bar{v}(k)\gamma^\nu v(k')$

(t-channel)

(b)

(b) Use the QED Feynman rules to show that amplitude can be written as:

$$i\mathcal{M} = \frac{ie^2}{t}A_1 - \frac{ie^2}{s}A_2$$

Products of spinors & gamma matrices.

Sol^n: • We already have done the computation for the t-channel in the $e\mu^- \to e\mu^-$ scattering from last week.

$$i\mathcal{M}_t^{*} = \frac{ie^2}{t}\underbrace{\eta_{\mu\nu}\,\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma^\nu v(k')}_{A_1}$$

• Using crossing symmetry & typical s-channel computation,

$$i\mathcal{M}_s = \frac{ie^2}{s}\underbrace{\eta_{\alpha\beta}\,\bar{v}(k)\gamma^\alpha u(p)\bar{u}(p')\gamma^\beta v(k')}_{A_2}$$

(c) ~~(e)~~ Calculate $|\mathcal{M}|^2$, in the high energy limit (neglect e^- mass)

[1] Use trace identities to show that the first term is given by

$$\frac{1}{4}\sum_{spin}|A_1|^2 = 2(u^2+s^2)$$

(Note: Same expression is obtained for the $e\mu$ scattering process)

→ As said, this is the exactly same computation for $e\mu^- \to e\mu^-$. (Feels redundant to go over the exact same algebra again)

[2] Calculate or use crossing-symmetry to get 2nd term.

For the s-channel diagram we can use crossing symmetry to get:

$$\frac{1}{4}\sum_{spin}|A_2| = 2(t^2+u^2)$$

[3] Use the identities $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$ & $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4(a\cdot b)$ to show that the two interference terms are given by

$$\frac{1}{4}\sum_{spin}A_1A_2^{*} = \frac{1}{4}\sum_{spin}A_2A_1^{*} = -2u^2$$

Lets now compute the cross terms:

$$\frac{1}{4}\sum_{spin} \mathcal{M}_s \mathcal{M}_t^* = \frac{1}{4}\sum_{spin}\left(\frac{ie^2}{s}\eta_{\alpha\beta}\bar{v}(k)\gamma^\alpha u(p)\bar{u}(p')\gamma^\beta v(k')\right)$$
$$\left(-\frac{ie^2}{t}\eta_{\mu\nu}\bar{v}(k')\gamma^\nu v(k)\bar{u}(p)\gamma^\mu u(p')\right)$$

$$= \frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{spin}\bar{v}_a(k)\gamma^\alpha_{ab}u_b(p)\bar{u}_c(p')\gamma^\beta_{cd}v_d(k')\gamma^\nu_{ef}v_f(k)$$
$$\bar{u}_g(p)\gamma^\mu_{gh}u_h(p')$$

$$= (-1)^{15}\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{spin}\gamma^\alpha_{ab}u_b(p)\bar{u}_g(p)\gamma^\beta_{cd}v_d(k')\bar{v}_e(k')\gamma^\nu_{ef}v_f(k)\bar{v}_a(k)$$
$$\gamma^\mu_{gh}u_h(p')\bar{u}_c(p')$$

↓ Drop mass terms & use: $\sum_{spin} u_s(p)\bar{u}_s(p) = \not{p}+m$, $\sum_{spin} v_s(p)\bar{v}_s(p) = \not{p}-m$

$$= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\not{k}_{fa}\gamma^\alpha_{ab}\not{p}_{bg}\gamma^\mu_{gh}\not{p}'_{hc}\gamma^\beta_{cd}\not{k}'_{de}\gamma^\nu_{ef}$$

Finishing the trace

$$= \cancel{-\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}Tr(\not{k}\gamma^\alpha\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu)}$$

$$= \frac{-e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}Tr\left[\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu\right]$$

- Now lets compute the trace: $Tr[\eta_{\alpha\beta}\gamma^\alpha\gamma^\beta] = \frac{1}{2}Tr[\eta_{\alpha\beta}\{\gamma^\alpha,\gamma^\beta\}] = 4$
  i) Eliminate the free gamma matrices (To do this we need to get them one besides the other)

$$\gamma^\alpha\not{p} = p_\beta\gamma^\alpha\gamma^\beta \;(2p^\alpha - \not{p}\gamma^\alpha)$$
$$= p_\beta\left(2\eta^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2p^\alpha - \not{p}\gamma^\alpha$$

Also, for any $\not{p}, \not{k}, p^\alpha, k^\alpha$

$$\not{p}\not{p} = p_\alpha p_\beta\gamma^\alpha\gamma^\beta = \frac{1}{2}p_\alpha p_\beta\{\gamma^\alpha,\gamma^\beta\} = \frac{1}{2}p_\alpha p_\beta 2\eta^{\alpha\beta} = p^2 = 0$$

ii) Use conservation of momentum, we will replace $k' = p + k - p'$

$$\frac{1}{4}\sum_{spin} M_s M_t^* = -\frac{e^4}{4st}\,\eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not k\gamma^\alpha\not p\gamma^\mu\not p'\gamma^\beta(\not p+\not k-\not p')\gamma^\nu\right]$$

$$= \frac{-e^4}{4st}\left(T_p + T_k - T_{p'}\right)$$

where

$$T_p = \eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not k\gamma^\alpha\not p\gamma^\mu\not p'\gamma^\beta\not p\gamma^\nu\right]$$

$$T_k = \text{———}\,"\,\text{———}\,\not k\,\text{—}$$

$$T_{p'} = \text{———}\,"\,\text{———}\,\not p'\,\text{—}$$

$$= \eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not p'\gamma^\nu\not k\gamma^\alpha\not p\gamma^\mu\not p'\gamma^\beta\right]$$

The last extra equality is from the cyclic prop. of Tr.
→ This enables us to see that T_k & $T_{p'}$ are related by cyclic substitution : $k \to p' \to p \to k$.

$$T_p = \eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not k\gamma^\alpha\not p\gamma^\mu\not p'\gamma^\beta\not p\gamma^\nu\right]$$

$$= \eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not k\gamma^\alpha(2p^\mu-\gamma^\mu\not p)\not p'(2p^\beta-\not p\gamma^\beta)\gamma^\nu\right]$$

$$= \cancel{4\,\mathrm{Tr}[\not k\not p\not p'\not p]} - \cancel{2\eta_{\alpha\beta}\mathrm{Tr}[\not k\gamma^\alpha\not p'\not p\gamma^\beta\not p]}$$
$$\cancel{- 2\eta_{\mu\nu}\mathrm{Tr}[\not k\not p\gamma^\mu\not p\not p'\gamma^\nu]} + \cancel{\eta_{\alpha\beta}\eta_{\mu\nu}\mathrm{Tr}[\not k\gamma^\alpha\not p'(2p^\beta-\gamma^\beta\not p)\not p]}$$

$$= 4\,\mathrm{Tr}\left[\not k\not p(2(p\cdot p')-\not p\not p')\right] - 2\eta_{\alpha\beta}\mathrm{Tr}\left[\not k\gamma^\alpha\not p'(2p^\beta-\gamma^\beta\not p)\not p\right]$$
$$- 2\eta_{\mu\nu}\mathrm{Tr}\left[\not k\not p(2p^\mu-\not p\gamma^\mu)\not p'\gamma^\nu\right] + \eta_{\alpha\beta}\eta_{\mu\nu}\mathrm{Tr}\left[\not k\gamma^\alpha\gamma^\mu(2(p\cdot p')-\not p'\not p)\not p\gamma^\beta\gamma^\nu\right]$$

Simplifying & using $p^2 = m^2 = 0$

$$T_p = 8(p\cdot p')\,\mathrm{Tr}(\not k\not p) - 4\,\mathrm{Tr}(\not k p^2\not p') - 4\,\mathrm{Tr}(\not k\not p\not p'\not p)$$
$$+ 2\eta_{\alpha\beta}\mathrm{Tr}(\not k\gamma^\alpha\not p'\gamma^\beta p^2) - 4\,\mathrm{Tr}[\not k\not p\not p'\not p]$$
$$+ 2\eta_{\mu\nu}\mathrm{Tr}[\not k p^2\gamma^\mu\not p'\gamma^\nu] + \eta_{\alpha\beta}\eta_{\mu\nu}2(p\cdot p')\mathrm{Tr}[\not k\gamma^\alpha\gamma^\mu\not p\gamma^\beta\gamma^\nu]$$

$$-\eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not{k}\gamma^{\alpha}\not{p}'\not{p}\not{k}\gamma^{\beta}\gamma^{\nu}\right]$$

$$= \cancel{32(p\cdot k)}\; 32(p\cdot p')(k\cdot p) - 4\,\mathrm{Tr}\left[\not{k}\not{p}\not{p}'\not{p}\right] - 4\,\mathrm{Tr}\left[\not{k}\not{p}\not{p}'\not{p}\right]$$

$$+2(p\cdot p')(k\cdot p)\; \cancel{}\; \eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not{k}\gamma^{\alpha}\gamma^{\mu}\not{p}\gamma^{\beta}\gamma^{\nu}\right]$$

$$= 32(p\cdot p')(k\cdot p) - 8\,\mathrm{Tr}\left[\not{k}\left(2(p\cdot p') - \not{p}'\not{p}\right)\not{p}\right]$$

$$+2(p\cdot\cancel{p}\,p')\,\underbrace{\eta_{\alpha\beta}\eta_{\mu\nu}\,\mathrm{Tr}\left[\not{k}\gamma^{\alpha}\gamma^{\mu}\not{p}\gamma^{\beta}\gamma^{\nu}\right]}_{16(k\cdot p)}$$

$$T_p = 32(p\cdot p')(k\cdot p) - 64(p\cdot p')(p\cdot k) + 32(p\cdot p')(k\cdot p)$$

$$= 0$$

$$T_k = \cdots = -32(p'\cdot k)(p\cdot k)$$

$$T_{p'} = -32(p\cdot p')(k\cdot p')$$

$$\Rightarrow \boxed{\frac{1}{4}\sum_{spin} \mathcal{M}_s \mathcal{M}_t^*} = -\frac{8e^4}{st}\left[(p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k)\right]$$

Similarly for second cross term.

$$\frac{1}{4}\sum_{spin} \mathcal{M}_t \mathcal{M}_s^* = \frac{-8e^4}{st}\left[(k\cdot k')(p\cdot k') - (k'\cdot p)(k\cdot p)\right]$$

Use kinematics

$p=(E,\bar{p})$, $k=(E,-\bar{p})$

$p'=(E,p')$, $k'=(E,-\bar{p}')$

Dropping masses & $p^2=E^2-m^2=E^2$

→ $p\cdot k = E^2+p^2 = 2E^2-m^2 = 2E^2$

→ $p'\cdot k = E^2+\bar{p}\cdot\bar{p}' = E^2+p^2\cos\theta = E^2(1+\cos\theta)$

→ $k'\cdot k =$ ——— ... ——— $= E^2(1-\cos\theta)$

→ $p\cdot p' =$ ——— ... ——— $= E^2(1-\cos\theta)$

→ $p\cdot k' =$ ——— ... ——— $= E^2(1+\cos\theta)$

→ $p'\cdot k' =$ ——— ... ——— $= 2E^2$

$t=(p-p')^2 = 2m^2-2p\cdot p' = -2E^2(1-\cos\theta)$

$s=(p+k)^2 = 4E^2$

$$\frac{1}{4}\sum|M|^2 = \frac{8e^4}{s^2}\left[(p\cdot p')(k\cdot k')+(p'\cdot k)(p\cdot k')\right]+\frac{8e}{t^2}\left[(p\cdot k')(p'\cdot k)+(p\cdot k)(p'\cdot k')\right]$$

$$-\frac{8e^2}{st}\left\{(p\cdot p')(k\cdot p')-(p'\cdot k)(p\cdot k)-(k'\cdot p)(k\cdot p)+(k\cdot k')(p\cdot k')\right.$$

Maple

$$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2 s}|M|^2=\frac{e^4}{64\pi^2 2E}\left[\frac{1}{2}(1+\cos^2\theta)+\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}-\frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right]$$