Mandelstam variables in center-of-mass (CM) frame

Problem Statement

It can be derived from first principles (or motivated with much less-rigorous arguments as in Schwartz chapter 5.3) that for a process like

$$e^+e^- \to \mu^+\mu^- \tag{1}$$

the cross section has a form,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{\text{CM}}^2}(1+\cos^2\theta) \tag{2}$$

in the COM frame.

(a). Work out the Lorentz-invariant quantities $s = (p_{e+} + p_{e^-})^2$, $t = (p_{\mu^-} - p_{e^-})^2$ and $u = (p_{\mu^+} - p_{e^-})^2$ in terms of E_{CM} and $\cos\theta$, while assuming $m_\mu = m_e = 0$.

(b). Derive a relationship between s, t and u.

(c). Rewrite $\frac{d\sigma}{d\Omega}$ in terms of s, t and u.

(d). Now assume m_μ and m_e are non-zero. Derive a relationship between s, t and u and the masses.

Solution.

Before even starting solving the individual questions, a good idea would be to have all the diagrams (for all the 3 channels) in front of us. Also it is important that we label everything properly.

Let,

$$(p_{e^-})_\mu = (p_1)_\mu, \quad (p_{e^+})_\mu = (p_2)_\mu, \quad (p_{\mu^-})_\mu = (p_3)_\mu, \quad (p_{\mu^+})_\mu = (p_4)_\mu \tag{3}$$

be our 4 vectors. (All the other quantities get the same labels for these particles)

Let us assume that the collision is happening on the (our-chosen) x-axis and the products are in the xy-plane (In the COM frame, we can choose this wlog). θ is the angle at which the products back-scatter. This gives us,

$$\begin{aligned}
(p_1)_\mu &= (E_1, \vec{p}_1) = (E_1, |\vec{p}_1|, 0, 0) \\
(p_2)_\mu &= (E_2, \vec{p}_2) = (E_2, |\vec{p}_2|, 0, 0) \\
(p_3)_\mu &= (E_3, \vec{p}_3) = (E_3, |\vec{p}_3|\cos\theta, |p_3|\sin\theta, 0) \\
(p_4)_\mu &= (E_4, \vec{p}_4) = (E_4, -|\vec{p}_4|\cos\theta, -|p_4|\sin\theta, 0)
\end{aligned} \tag{4}$$

In the CM reference frame,

$$\vec{p}_1 = -\vec{p}_2 \implies |\vec{p}_1| = |\vec{p}_2| \tag{5}$$
$$\vec{p}_3 = -\vec{p}_4 \implies |\vec{p}_3| = |\vec{p}_4| \tag{6}$$

To even simplify further the notation, let $|\vec{p}_1| = p$ and $|\vec{p}_3| = k$. This gives us,

$$p \equiv |\vec{p}_1| = |\vec{p}_2| \tag{7}$$
$$k \equiv |\vec{p}_3| = |\vec{p}_4| \tag{8}$$

We define the CM energy by E_{CM} using the following

$$E_1 + E_2 = E_3 + E_4 = E_{\text{CM}} \tag{9}$$

Where E_{CM} is the center-of-mass frame. The energies are given by,

$$\begin{aligned} E_1 &= \sqrt{m_1^2 + |\vec{p}_1|^2} \overset{m_1=0}{=} |\vec{p}_1| = p \\ E_2 &= \sqrt{m_2^2 + |\vec{p}_2|^2} \overset{m_2=0}{=} |\vec{p}_2| = p \\ E_3 &= \sqrt{m_3^2 + |\vec{p}_3|^2} \overset{m_3=0}{=} |\vec{p}_3| = k \\ E_4 &= \sqrt{m_4^2 + |\vec{p}_4|^2} \overset{m_4=0}{=} |\vec{p}_4| = k \end{aligned} \tag{10}$$

From this E_{CM} can be written as,

$$E_{\text{CM}} = 2k = 2p \tag{11}$$
$$\implies k = p = \frac{E_{\text{CM}}}{2} \tag{12}$$

which helps us write down the all of our 4 vectors using just one variable E_{CM} (at least for the zero particle masses scenario) by plugging in eq.(10), eq.(12) in eq.(61),

$$\boxed{\begin{aligned} (p_1)_\mu &= (p, p, 0, 0) = \frac{E_{CM}}{2}(1, 1, 0, 0) \\ (p_2)_\mu &= (p, -p, 0, 0) = \frac{E_{CM}}{2}(1, -1, 0, 0) \\ (p_3)_\mu &= (k, k\cos\theta, k\sin\theta, 0) = \frac{E_{CM}}{2}(1, \cos\theta, \sin\theta, 0) \\ (p_4)_\mu &= (k, -k\cos\theta, -k\sin\theta) = \frac{E_{CM}}{2}(1, -\cos\theta, -\sin\theta, 0) \end{aligned}} \tag{13}$$

Solution (a).

Now that we have set up all the variables in terms of $E_{\text{CM}}, \cos\theta$, we can calculate s, t, u as a function of those variables (as asked in the question) quite easily.

Starting with s,

$$s = ((p_1)_\mu + (p_2)_\mu)^2 = \left(\frac{E_{\text{CM}}}{2}(1,1,0,0) + \frac{E_{\text{CM}}}{2}(1,-1,0,0)\right)^2 \tag{14}$$

$$= \left(\frac{E_{CM}}{2}(2,0,0,0)\right)^2 \tag{15}$$

$$= \frac{E_{\text{CM}^2}}{4}(4-0) \tag{16}$$

$$= E_{\text{CM}}^2 \tag{17}$$

$$\boxed{s = E_{CM}^2} \tag{18}$$

Now, similarly we calculate t,

$$t = ((p_1)_\mu - (p_3)_\mu)^2 = \left(\frac{E_{\text{CM}}}{2}(1,1,0,0) - \frac{E_{\text{CM}}}{2}(1,\cos\theta,\sin\theta,0)\right)^2 \tag{19}$$

$$= \left(\frac{E_{\text{CM}}}{2}(0,1-\cos\theta,-\sin\theta,0)\right)^2 \tag{20}$$

$$= \frac{E_{\text{CM}}^2}{4}(0^2 - (1-\cos\theta)^2 - (-\sin\theta)^2 - 0) \tag{21}$$

$$= \frac{E_{\text{CM}}^2}{4}(-1 + 2\cos\theta \underbrace{-\cos^2\theta - \sin^2\theta}_{-1}) \tag{22}$$

$$= \frac{E_{\text{CM}}^2}{4}(-2 + 2\cos\theta) \tag{23}$$

$$= \frac{E_{\text{CM}}^2}{2}(-1 + \cos\theta) \tag{24}$$

$$= -\frac{E_{\text{CM}}^2}{2}(1 - \cos\theta) \tag{25}$$

where we did the last step of rearrangement because we are used to seeing cross-sections in terms of $(1 \pm \cos\theta)$.

$$\boxed{t = -\frac{E_{\text{CM}}^2}{2}(1 - \cos\theta)} \tag{26}$$

Finally, for u,

$$u = ((p_1)_\mu - (p_4)_\mu)^2 = \left(\frac{E_{\text{CM}}}{2}(1,1,0,0) - \frac{E_{\text{CM}}}{2}(1,-\cos\theta,-\sin\theta,0)\right)^2 \tag{27}$$

$$= \left(\frac{E_{\text{CM}}}{2}(0,1+\cos\theta,\sin\theta,0)\right)^2 \tag{28}$$

$$= \frac{E_{\text{CM}}^2}{4}(0^2 - (1+\cos\theta)^2 - (\sin\theta)^2 - 0) \tag{29}$$

$$= \frac{E_{\text{CM}}^2}{4}(-1 - 2\cos\theta \underbrace{- \cos^2\theta - \sin^2\theta}_{-1}) \tag{30}$$

$$= \frac{E_{\text{CM}}^2}{4}(-2 - 2\cos\theta) \tag{31}$$

$$= \frac{E_{\text{CM}}^2}{2}(-1 - \cos\theta) \tag{32}$$

$$\boxed{u = -\frac{E_{\text{CM}}^2}{2}(1+\cos\theta)} \tag{33}$$

Solution (b).

A common relation between s, t, u can be derived by adding them,

$$s + t + u = E_{\text{CM}}^2 - \frac{E_{\text{CM}}^2}{2}(1-\cos\theta) - \frac{E_{\text{CM}}^2}{2}(1+\cos\theta) \tag{34}$$

$$= E_{\text{CM}}^2 - \frac{E_{\text{CM}}^2}{2} - \frac{E_{\text{CM}}^2}{2} + \frac{E_{\text{CM}}^2}{2}(\cos\theta - \cos\theta) \tag{35}$$

$$= 0 \tag{36}$$

$$\boxed{s + t + u = 0} \tag{37}$$

Solution (c).

We want to rewrite the given cross section in the question in terms of the Madelstam variables.

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{\text{CM}}^2}(1+\cos^2\theta) \tag{38}$$

From what we have found, we can use,

$$E_{\text{CM}}^2 = s \tag{39}$$

$$(1 + \cos^2\theta) = \frac{2(t^2 + u^2)}{s^2} \tag{40}$$

The second equation we find by a quick mathematica computation as seen in fig.(1)

I have used $E_{CM}{}^2 = s$ and $\cos\theta = y$

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In[31]:= Reduce[{t == -s/2 (1 - y), u == -s/2 (1 + y), s + t + u == 0}, y]

Out[31]= (s == -t - u && t + u ≠ 0 && y == (-t + u)/(t + u)) || (u == 0 && t == 0 && s == 0)
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This gives: $\cos\theta = \frac{-t+u}{t+u}$

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In[41]:= y = (-t + u)/(t + u)

Out[41]= (-t + u)/(t + u)

In[46]:= 1 + y^2 // Simplify

Out[46]= 2 (t^2 + u^2)/(t + u)^2
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★ Using $(t+u) = s$, we have the following relation: $(1 + \mathrm{Cos}^2\,\theta) = \frac{2(t^2+u^2)}{s^2}$

Figure 1: A quick mathematica computation to find the relation between $\cos\theta$ and s, t, u. The equation holds true as long as $u + t \neq 0$, i.e. $E_{\mathrm{CM}} \neq 0$, which is something we can safely assume.

The rearranged cross section is,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 s^3}(t^2 + u^2)} \tag{41}$$

Solution (d).

Here, we just want to find s, t and u for the case where we reintroduce the masses

$$m_{e^-} = m_{e^+} = m_1, \quad m_{\mu^-} = m_{\mu^+} = m_2 \tag{42}$$

Actually, let us just work where all the masses are different, m_1, m_2, m_3 and m_4. Just like last time, we add the three Mandelstam variables,

$$s + t + u = \left((p_1)_\mu + (p_2)_\mu\right)^2 + \left((p_1)_\mu - (p_3)_\mu\right)^2 + \left((p_1)_\mu - (p_4)_\mu\right)^2 \tag{43}$$

Just for this calculation's sake, I am skipping the 4-vector indices, so $(p_1)_\mu \equiv p_1$. That means $p_1^2 = (p_1)_\mu (p_1)^\mu$ and $p_1 \cdot p_2 = (p_1)_\mu (p_2)^\mu$

Calculating the individual squared terms,

$$\begin{aligned} p_1^2 &= (m_1^2 + |\vec{p}_1|^2 - |\vec{p}_1^2|) &&= m_1^2 \\ p_2^2 &= \dots &&= m_2^2 \\ p_3^2 &= \dots &&= m_3^2 \\ p_4^2 &= \dots &&= m_4^2 \end{aligned} \tag{44}$$

Again expanding the sum of the three Mandelstam variables,

$$s + t + u = p_1^2 + p_2^2 + 2p_1 \cdot p_2 + p_1^2 + p_3^2 - 2p_1 \cdot p_3 + p_1^2 + p_4^2 - 2p_1 \cdot p_4 \tag{45}$$

$$= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (p_2 - p_3 - p_4) \tag{46}$$

Recall that energy conservation plus momentum conservation is both built into conservation of 4-momentum (The zeroth component takes care of the energy and the i'th components takes care of momentum conservation). So we can use,

$$\begin{aligned} p_1 + p_2 &= p_3 + p_4 \\ p_1 &= -p_2 + p_3 + p_4 \\ -p_1 &= p_2 - p_3 - p_4 \end{aligned} \tag{47}$$

The last equation here can be plugged into the last term of the previous equation giving us,

$$s + t + u = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 - 2p_1 \cdot p_1 \tag{48}$$

$$= p_1^2 + p_2^2 + p_3^2 + p_4^2 \tag{49}$$

$$= m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{50}$$

This gives us such an immensely important result used frequently in particle physics,

$$\boxed{s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2} \tag{51}$$

Comment

Let us see how do the 4-vectors look in the massive case,

$$m_{e^-} = m_{e^+} = m_1, \quad m_{\mu^-} = m_{\mu^+} = m_2 \tag{52}$$

We will now use eq.(10),

$$\begin{aligned} E_1 &= \sqrt{m_1^2 + |\vec{p}_1|^2} = \sqrt{m_1^2 + p^2} \\ E_2 &= \sqrt{m_2^2 + |\vec{p}_1|^2} = \sqrt{m_1^2 + p^2} \\ E_3 &= \sqrt{m_3^2 + |\vec{p}_2|^2} = \sqrt{m_2^2 + k^2} \\ E_4 &= \sqrt{m_4^2 + |\vec{p}_2|^2} = \sqrt{m_2^2 + k^2} \end{aligned} \tag{53}$$

Now we write E_{CM},

$$E_{\text{CM}} = E_1 + E_2 \tag{54}$$

$$= 2\sqrt{m_1^2 + p^2} \tag{55}$$

$$E_{\text{CM}} = E_3 + E_4 \tag{56}$$

$$= 2\sqrt{m_2^2 + k^2} \tag{57}$$

$$\frac{E_{\text{CM}}}{2} = \sqrt{m_1^2 + p^2} = \sqrt{m_2^2 + k^2} \tag{58}$$

$$= E_1 = E_2 = E_3 = E_4 \tag{59}$$

The first component of all 4 vectors will remain the same, but we won't be able to pull out the $\frac{E_{\text{CM}}}{2}$ factor outside the 4-vectors as we did in the massless case. This is the furthest simplification we can make in the $m_1 = m_2, m_3 = m_4$ massive case,

$$\begin{aligned} (p_1)_\mu &= (E_1, \vec{p}_1) = \left(\frac{E_{\text{CM}}}{2}, p, 0, 0\right) \\ (p_2)_\mu &= (E_2, \vec{p}_2) = \left(\frac{E_{\text{CM}}}{2}, p, 0, 0\right) \\ (p_3)_\mu &= (E_3, \vec{p}_3) = \left(\frac{E_{\text{CM}}}{2}, k\cos\theta, k\sin\theta, 0\right) \\ (p_4)_\mu &= (E_4, \vec{p}_4) = \left(\frac{E_{\text{CM}}}{2}, -k\cos\theta, -k\sin\theta, 0\right) \end{aligned} \tag{60}$$

If all the 4 masses are different $m_1 \neq m_2 \neq m_3 \neq m_4$ (all non-zero), then the following is the most simplification we can achieve,

$$\begin{aligned}
(p_1)_\mu &= (E_1, \vec{p}_1) = (E_1, |\vec{p}_1|\,, 0, 0) = (E_1, p, 0, 0) \\
(p_2)_\mu &= (E_2, \vec{p}_2) = (E_2, |\vec{p}_2|\,, 0, 0) = (E_2, p, 0, 0) \\
(p_3)_\mu &= (E_3, \vec{p}_3) = (E_3, |\vec{p}_3| \cos\theta, |p_3| \sin\theta, 0) = (E_3, p\cos\theta, p\sin\theta, 0) \\
(p_4)_\mu &= (E_4, \vec{p}_4) = (E_4, -|\vec{p}_4| \cos\theta, -|p_4| \sin\theta, 0) \\
&= (E_4, -k\cos\theta, -k\sin\theta, 0)
\end{aligned} \tag{61}$$

If some of the masses are zero, you can start from here and *might* be able to represent again in terms of E_{CM}.