6.2 (Noether's theorem, symmetries and conserved charges)

Recall the Noether's theorem, which says that any continuous symmetry of the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ results in a conserved current. That is, if the varitation of the field is given by $\delta\phi$ (i.e. $\phi \to \phi + \delta\phi$)and the Lagrangian changes at most by a complete derivative $\delta\mathcal{L} = \partial_\mu F^\mu$, then

$$\partial_\mu j^\mu = 0, \qquad \text{where} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu$$

(a) Prove Noether's theorem

Solution. (A very good look read would be chapter 3 of Schwartz's textbook)

To derive the Euler-Lagrange equations, we start by varying the action. Similarly, in this case, let us start by varying the Lagrangian density,

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)$$

Use the fact that

$$\begin{aligned}\delta(\partial_\mu \phi) &= \partial_\mu \phi' - \partial_\mu \phi \\ &= \partial_\mu(\phi + \delta\phi) - \partial_\mu \phi \\ &= \partial_\mu(\delta\phi)\end{aligned}$$

Using this in $\delta\mathcal{L}$,

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)$$

We can now use the fact that,

$$\begin{aligned}\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi \\ \to \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) &= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi\end{aligned}$$

Plugging the equation arrowed above into $\delta\mathcal{L}$,

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) \\ &\equiv \partial_\mu F^\mu\end{aligned}$$

In the $\delta\mathcal{L}$ we can factorize the $\delta\phi$ in the first two terms and then the terms in the bracket are just the Euler-Lagrange equations

$$\delta\mathcal{L} = \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right)}_{=0}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) \equiv \partial_\mu F^\mu$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu\right) = 0$$

where we can define our **Noether current** as,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu$$

Conclusion : After assuming our transformation, the Lagrangian stays invariant up-to a total derivative, then it implies that we have a conserved current given by j^μ as in the expression above.

□

(b) Show that a conserved charge $Q = \int d^3x\, j^0$ satisfies $\partial_t Q = 0$.

Solution.
Assuming the result of (a),

$$\partial_\mu j^\mu = 0$$

Contracting the indices with Minkwoski metric $(- + ++)$

$$\begin{aligned}
\eta^{\mu\nu}\partial_\mu j_\nu &= 0 \\
-\partial_0 j_0 + \partial_i j_i &= 0 \\
\partial_t j_0 &= \vec{\nabla}\cdot\vec{j} \\
\int_V d^3x\, \partial_t j_0 &= \int_V d^3x\, \vec{\nabla}\cdot\vec{j} \\
\partial_t \int_V d^3x\, j_0 &= \int_{\partial V} d\vec{\sigma}\cdot\vec{j} \\
\partial_t Q &= 0 \qquad \text{(Surface terms vanish for } V\to\infty)
\end{aligned}$$

□

(c) Show that the symmetry under the infinitesimal translation

$$x^\mu \to x^\mu + \varepsilon^\mu, \qquad \phi(x) \to \phi(x-\varepsilon)$$

results in 4-conserved currents $j_0^\mu, j_1^\mu, j_2^\mu, j_3^\mu$, also known as the energy-momentum tensor $T^{\mu\nu}$.
Hint: Taylor expand $\phi(x-\varepsilon)$ and $\mathcal{L}(x-\varepsilon)$ to find $\delta\phi$ and F^μ.

Solution.
The goal of this problem is to show that, translational symmetry results in 4 conserved currents.
Taylor expand the infinitesimal translations given to us

$$\begin{aligned}
\phi(x-\varepsilon) &= \phi(x-\varepsilon)\big|_{\varepsilon=0} + \frac{\partial\phi(x-\varepsilon)}{\partial\varepsilon^\mu}\Big|_{\varepsilon=0}\cdot\varepsilon^\mu \\
&= \phi(x) + \left[\frac{\partial\phi(x-\varepsilon)}{\partial(x^\nu-\varepsilon^\nu)}\frac{\partial(x^\nu-\varepsilon^\nu)}{\partial\varepsilon^\mu}\right]_{\varepsilon=0}\cdot\varepsilon^\mu \\
&= \phi(x) + \left[\frac{\partial\phi(x)}{\partial(x^\nu)}\left(-\delta^\nu_\mu\right)\right]_{\varepsilon=0}\cdot\varepsilon^\mu \\
&= \phi(x) - \partial_\nu\phi\,\delta^\nu_\mu\,\varepsilon^\mu \\
\phi(x-\varepsilon) &= \phi(x) - \varepsilon^\mu\partial_\mu\phi \\
\text{(Comparing with)} &\equiv \phi(x) + \delta\phi
\end{aligned}$$

we get,

$$\delta\phi = -\varepsilon^\mu\partial_\mu\phi$$

Similarly, for $\delta\mathcal{L}$

$$\delta\mathcal{L} = -\varepsilon^m\partial_\mu\mathcal{L}(x) \stackrel{!}{=} \partial_\mu F^\mu$$

As ε^μ is just a constant object,

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu\left(-\varepsilon^\mu\mathcal{L}(x)\right) \\
\Rightarrow F^\mu &= -\varepsilon^\mu\mathcal{L}(x)
\end{aligned}$$

Recall the following expression from the previous part

$$
\begin{aligned}
j^{\mu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\delta\phi - F^{\mu} \\
&= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(-\varepsilon^{\nu}\partial_{\nu}\phi\right) + \varepsilon^{\mu}\mathcal{L}(x) \\
&= -\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\varepsilon^{\nu}\partial_{\nu}\phi\right) + \varepsilon^{\mu}\mathcal{L}(x) \\
&= -\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\partial_{\nu}\phi\right)\varepsilon^{\nu} + \eta^{\mu\nu}\varepsilon_{\nu}\mathcal{L}(x)
\end{aligned}
$$

Using the fact that $a^{\mu}b_{\mu} = a_{\mu}b^{\mu}$,

$$
\begin{aligned}
&= -\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\partial^{\nu}\phi\right)\varepsilon_{\nu} + \eta^{\mu\nu}\varepsilon_{\nu}\mathcal{L}(x) \\
&= \left[-\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\partial^{\nu}\phi\right) + \eta^{\mu\nu}\mathcal{L}(x)\right]\varepsilon_{\nu} \\
&= \left[-\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\partial_{\rho}\phi\right)\eta^{\rho\nu} + \eta^{\mu\nu}\mathcal{L}(x)\right]\varepsilon_{\nu}
\end{aligned}
$$

This shows that we have one current for each direction of ε_{ν} (because it has 4 components), then we have one for each direction,

$$
T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\left(\partial_{\rho}\phi\right)\eta^{\rho\nu} + \eta^{\mu\nu}\mathcal{L}(x)
$$

Even if we started with one index μ, we end up with 4 different ν for each of them. Also, under the exchange or μ and ν it is symmetric.Giving us 6 independent components.

□

(d) Show that in the infinitesimal Lorentz transformation $\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$, the infinitesimal parameter is anti-symmetric, i.e.

$$
\omega^{\mu\nu} = -\omega^{\nu\mu}
$$

Solution.
We have

$$
\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \delta\Lambda
$$

We also know

$$
\begin{aligned}
\eta_{\rho\sigma} &= \Lambda^{\mu}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}_{\sigma} \\
&= \left(\delta^{\mu}_{\rho} + \omega^{\mu}_{\rho}\right)\eta_{\mu\nu}\left(\delta^{\nu}_{\sigma} + \omega^{\nu}_{\sigma}\right) \\
&= \delta^{\mu}_{\rho}\eta_{\mu\nu}\delta^{\nu}_{\sigma} + \delta^{\mu}_{\rho}\eta_{\mu\nu}\omega^{\nu}_{\sigma} + \omega^{\mu}_{\rho}\eta_{\mu\nu}\delta^{\nu}_{\sigma} + \underbrace{\omega^{\mu}_{\rho}\eta_{\mu\nu}\omega^{\nu}_{\sigma}}_{\mathcal{O}(\omega^2)} \\
&= \eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} \\
&\stackrel{!}{=} \eta_{\rho\sigma}
\end{aligned}
$$

The only way that we can replace $\stackrel{!}{=}$ with $=$ is if,

$$
\omega_{\sigma\rho} = -\omega_{\rho\sigma}
$$

□

(e) Consider the action of the free classical real scalar field

$$
S = \int d^4x\, \mathcal{L} = \int d^4x \left(-\frac{1}{2}\partial_{\mu}\phi(x)\,\partial^{\mu}(x) - \frac{1}{2}m^2\phi(x)\,\phi(x)\right)
$$

Find the corresponding energy-momentum tensor to the infinitesimal Lorentz transformation.

Solution.
We have

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi(x)\,\partial^\mu \phi(x) - \frac{1}{2}m^2 \phi(x)\,\phi(x)$$

We need to do an infinitesimal Lorentz transformation, which would be $x^\nu \to x^\mu$

$$\begin{aligned} x^\mu &= \Lambda^\mu_\nu x^\nu = \left(\delta^\mu_\nu + \omega^\mu_\nu\right) x^\nu \\ \text{with} \quad \omega^{\mu\nu} &= -\omega^{\nu\mu} \end{aligned}$$

The energy momentum tensor upon Lorentz transformation is

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial^\nu \phi + \eta^{\mu\nu}\mathcal{L}$$

In a nutshell, we just have to compute the above expression for the given Lagrangian. With slight change of notation we can write,

$$\mathcal{L} = -\frac{1}{2}\eta^{\rho\sigma}\,\partial_\rho \phi\,\partial_\sigma \phi - \frac{1}{2}m^2\phi^2$$

Calculating

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} &= -\frac{1}{2}\eta^{\rho\sigma}\left(\partial_\rho \phi\,\delta^\mu_\sigma + \delta^\mu_\rho\,\partial_\sigma \phi\right) \\ &= -\frac{1}{2}\,\partial^\mu \phi(x) - \frac{1}{2}\partial^\mu \phi(x) \\ &= -\partial^\mu \phi(x) \end{aligned}$$

Giving us,

$$T^{\mu\nu} = +\partial^\mu \phi(x)\,\partial^\nu \phi(x) + \eta^{\mu\nu}\left(-\frac{1}{2}\partial_\mu \phi(x)\,\partial^\mu \phi(x) - \frac{1}{2}m^2 \phi(x)\,\phi(x)\right)$$

writing it in a more convenient way (grouping terms, arranging indices)

$$\begin{aligned} T^{\mu\nu} &= \partial_\rho \phi(x)\,\partial_\sigma \phi(x)\,\eta^{\rho\mu}\eta^{s\nu} - \frac{1}{2}\left(\partial_\rho \phi(x)\,\partial_\sigma \phi(x)\right)\eta^{\mu\nu}\eta^{\sigma\rho} - \frac{1}{2}\eta^{\mu\nu}m^2\phi^2 \\ &= \partial_\rho \phi\,\partial_\sigma \phi\left[\eta^{\rho\mu}\eta^{\sigma\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\rho}\right] - \frac{1}{2}\eta^{\mu\nu}m^2\phi^2 \end{aligned}$$

For this to hold true for Lorentz transformations, $\mathcal{L}$ must be invariant up to a total derivative. Even if we have found an expression for $T^{\mu\nu}$, it will only be conserved it satisfies invariance upto a total derivative.

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L}$$

the transformation that we want is,

$$\begin{aligned} x &\to \Lambda x \\ \mathcal{L} \to \mathcal{L}'(x) &= \mathcal{L}\left(\Lambda^{-1}x\right) \\ &= \mathcal{L}\left(\delta^\mu_\nu x^\nu - \omega^\mu_\nu x^\nu\right) \end{aligned}$$

Taylor expand the last parenthesis as a function of ω^μ_ν around $\omega^\mu_\nu = 0$,

$$\begin{aligned} \mathcal{L}\left(\Lambda^{-1}x\right) &\simeq \mathcal{L}\left(\Lambda^{-1}x\right)|_{\omega=0} + \frac{\partial \mathcal{L}}{\partial \omega^{\mu\nu}}|_{\omega=0}\omega^\mu_\nu \\ &= \mathcal{L}\left(\delta^\mu_\nu x^\nu - \omega^\mu_\nu x^\nu\right)|_{\omega=0} + \frac{\partial \mathcal{L}\left(\Lambda^{-1}x\right)}{\partial\left(\Lambda^{-1}x\right)}|_{\omega=0}\frac{\partial\left(\Lambda^{-1}x\right)}{\partial \omega^\mu_\nu}|_{\omega=0}\omega^\mu_\nu \\ &= \mathcal{L}\left(x^\mu\right) + \partial_\mu\left(\mathcal{L}(x)\right)\left(-x^\nu\right)\omega^\mu_\nu \\ \Rightarrow \delta\mathcal{L} &= -x^\nu\omega^\mu_\nu\partial_\mu\mathcal{L} \end{aligned}$$

The transformation parameter is a constant so we can pull it inside the ∂_μ, also recall that it is antisymmetric

$$\begin{aligned}\Rightarrow \delta\mathcal{L} &= -x^\nu \omega^\mu_\nu \partial_\mu \mathcal{L} \\ &= -x^\nu \partial_\mu \left(\omega^\mu_\nu \mathcal{L}\right) \\ &= -\partial_\mu \left(x^n \omega^\mu_\nu \mathcal{L}\right) + \underbrace{\omega^\mu_\nu}_{\text{Anti-sym}} \mathcal{L} \underbrace{\partial_\mu x^\nu}_{\text{Symmetric}}\end{aligned}$$

Switching the indices for symm+anti symm, we will get a zero from the sum, giving us

$$\delta\mathcal{L} = -\partial_\mu \left(x^\nu \omega^\mu_\nu \mathcal{L}\right)$$

which is a total derivative.
Using what we found in the start of the problem,

$$F^\mu = -x^\nu \omega^\mu_\nu \mathcal{L}$$

↓ have to finish

(f) Construct and identify the corresponding Noether charges associated with rotations ω^{ij} and boosts ω^{0i}.

Solution.
Recall

$$T^{\mu\nu} = \partial_\rho\phi\,\partial_\sigma\phi\left[\eta^{\rho\mu}\eta^{\sigma\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\rho}\right] - \frac{1}{2}\eta^{\mu\nu} m^2\phi^2$$

Currents are given by

$$\begin{aligned}j^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi - F^\mu \\ &= -\partial^\mu\phi\,\delta\phi + x^\nu\omega^\mu_\nu\mathcal{L}\end{aligned}$$

with $\delta\phi = -\omega^{\sigma\nu}x_\nu\partial_\sigma\phi$.