12.1 Gauge fields

Consider Lagrangian for electromagnetic field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

(a) Find the equation of motion for the photon field $A_\mu(x)$

Solution.
We have the action

$$S = -\frac{1}{4}\int_x F_{\mu\nu}F^{\mu\nu}$$

where

$$F = \partial_\mu A_\nu - \partial_\nu A_\mu$$

giving us

$$\begin{aligned}
S &= -\frac{1}{4}\int_x (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}\int_x (\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu - \partial_\nu A_\mu\,\partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) \\
&= -\frac{1}{4}\int_x (\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu - \partial_\mu A_\nu\,\partial^\nu A^\mu + \partial_\mu A_\nu \partial^\mu A^\nu) \\
&= -\frac{1}{4}\int_x ((\partial_\mu A_\nu\,\partial^\mu A^\nu + \partial_\mu A_\nu \partial^\mu A^\nu) - (\partial_\mu A_\nu\,\partial^\nu A^\mu + \partial_\mu A_\nu\,\partial^\nu A^\mu)) \\
&= -\frac{1}{4}\int_x (2(\partial_\mu A_\nu\,\partial^\mu A^\nu) - 2\,(\partial_\mu A_\nu\,\partial^\nu A^\mu)) \\
&= -\frac{1}{2}\int_x (\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu)
\end{aligned}$$

This was just algebraic manipulation till now. Now, for QFT computations we can assume the fields have compact suppport and we can ignore boundary terms. So, now using integration by parts,

$$\begin{aligned}
S &= -\frac{1}{2}\int_x (\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu) \\
&= \frac{1}{2}\int_x (A_\nu \partial_\mu \partial^\mu A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu) \\
&= \frac{1}{2}\int_x A_\nu\left(\partial_\mu \partial^\mu (\delta^\nu_\mu A^\mu) - \partial_\mu \partial^\nu A^\mu\right) \\
S &= \frac{1}{2}\int_x A_\nu\left(\partial_\mu \partial^\mu \delta^\nu_\mu - \partial_\mu \partial^\nu\right) A^\mu
\end{aligned}$$

Using Euler-Lagrange equations,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\nu)}\right) \\
\frac{1}{2}\left(\partial_\mu \partial^\mu \delta^\nu_\mu - \partial_\mu \partial^\nu\right) A^\mu &= 0 \\
\left(\partial_\mu \partial^\mu \delta^\nu_\mu - \partial_\mu \partial^\nu\right) A^\mu &= 0
\end{aligned}$$

We get the zero on the RHS because the Lagrangian density does not depend on derivatives of A_ν.
We can now go to momentum space for A^μ. Like always, every $\partial_\mu \to p_\mu$ and $\partial^\nu \to p^\nu$, giving us,

$$\begin{aligned}
\left(p_\mu p^\mu \delta^\nu_\mu - p_\mu p^\nu\right) A^\mu &= 0 \\
\left(p^2 \delta^\nu_\mu - p_\mu p^\nu\right) A^\mu &= 0
\end{aligned}$$

This is the equation of motion.

(b) Show that the Lagrangian density is invariant under gauge transformations.

Solution.
Gauge transformation (by the derivative of a spacetime dependent gauge fixing parameter $\alpha(x)$)

$$A_\mu \to A_\mu + \frac{1}{e}\left(\partial_\mu \alpha(x)\right)$$

Using this, we need to see how does $F_{\mu\nu}$ transform,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ \text{(gauge tranformation)} &= \partial_\mu\left(A_\nu + \frac{1}{e}\left(\partial_\nu \alpha(x)\right)\right) - \partial_\nu\left(A_\mu + \frac{1}{e}\left(\partial_\mu \alpha(x)\right)\right) \\ &= \underbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)}_{=F_{\mu\nu}} + \frac{1}{e}\underbrace{\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)}_{=0}\alpha(x) \\ &= F_{\mu\nu} \end{aligned}$$

↓ have to finish

(c) To solve the equation of motion, one has to restrict the gauge fields A^μ to satisfy the gauge fixing condition $\partial_\mu A^\nu = 0$. In momentum space it can be written as $p_\mu A^\nu = 0$. It is useful to introduce the transverse operator

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$$

Show that

$$\hat{A}^\mu(p) = P^{\mu\nu} A_\mu(p)$$

fulfills the gauge condition.

(d) Find the propagator $\Delta^{\mu\nu}(x-y)$ as a solution of the following equation

$$\left[-\eta_{\mu\nu}\partial_\alpha\partial^\alpha + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]\Delta^{\nu\rho}(x-y) = \delta^\rho_\mu \delta^4(x-y)$$

where ξ is the gauge fixing parameter. In particular the choice $\xi = 0$ corresponds to Landau gauge, $\partial_\mu A^\mu = 0$. Why cannot you solve the above equation without gauge fixing?
Hint : Fourier transform the equation

Solution.
Start with the equation we have,

$$\left[-\eta_{\mu\nu}\partial_\alpha\partial^\alpha + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]\Delta^{\nu\rho}(x-y) = \delta^\rho_\mu \delta^4(x-y)$$

Now, go to momentum space by taking the Fourier transform using,

$$\begin{aligned} \Delta^{\nu\rho}(x-y) &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \Delta^{\nu\rho}(p) \\ \delta^4(x-y) &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \end{aligned}$$

giving us,

$$\left[\eta_{\mu\nu}p^2 - \left(1 - \frac{1}{\xi}\right)p_\mu p_\nu\right]\Delta^{\nu\rho}(p) = \delta^\rho_\mu$$

we will be solving this equation, we need an ansatz for this

$$\Delta^{\mu\nu}(p) = A(p)\,\eta^{\mu\nu} + B(p)\,p^{\mu}p^{\nu}$$