~~QFT~~.P.10.2 (Trace identities)

QFT. Ex Sheet 10
Prob. 10.2.
$\boxed{\text{Pg 1.}}$

Given: → $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{n\times n}$ where $\mu,\nu = 0,1,2,\dots,d-1$

→ Dimension of γ-matrices is $\begin{cases} \xrightarrow{d\,\text{even}} n = 2^{d/2} \\ \xrightarrow[d\,\text{odd}]{} n = 2^{(d-1)/2} \end{cases}$

→ Without using any representation, prove the following identities in $d=4$, $n = 2^2 = 4$.

Also → $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu\gamma^\mu$, $\Sigma^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu]$

(a) $\{\gamma^\mu, \gamma^5\} = 0$

→ $\{\gamma^\mu, -i\gamma^0\gamma^1\gamma^2\gamma^3\} = -i\{\gamma^\mu, \gamma^0\gamma^1\gamma^2\gamma^3\}$

$= -i[\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu]$

For $\mu = 0$

$= -i[\underbrace{\gamma^0\gamma^0}_{1}\gamma^1\gamma^2\gamma^3 + \gamma^0\underbrace{\gamma^1}_{-1}\underbrace{\gamma^2}_{-1}\underbrace{\gamma^3}_{-1}\gamma^0]$

$= -i[\gamma^1\gamma^2\gamma^3 + (-1)^3\gamma^1\gamma^2\gamma^3\underbrace{\gamma^0\gamma^0}_{1}]$

$= -i[\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3] = 0$

↓ can similarly be proved for $\mu = 1,2,3$.

For $\mu = j$

$= -i[\gamma^j\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^j]$

OR.

$\{\gamma^\mu, \gamma^5\} = -i[\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu]$ → If we move it 4 time we get $(-1)^4$, but, one of the 4 it will commute with → $(-1)^3 = -1$

$= -i[\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3] = 0.$

(b) $(\gamma 5)^2 = \mathbb{I}_{4\times4}.$

$$\rightarrow (\gamma^5)^2 = -i\,\gamma^0\gamma^1\gamma^2\gamma^3\,(-i)\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= -(-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = -(-1)^3(-1)^2\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2 \gamma^3\gamma^2\gamma^3$$

$$= -\underbrace{\underset{-1}{(-1)^3}\,\underset{1}{(-1)^2}\,\underset{-1}{(-1)^1}}_{-1}\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3 = -\underset{-\mathbb{1}}{(\gamma^0)^2}\,\underset{\mathbb{1}}{(\gamma^1)^2}\,\underset{\mathbb{1}}{(\gamma^2)^2}\,\underset{\mathbb{1}}{(\gamma^3)^2}$$

$$= \mathbb{1}$$

(c) $\mathrm{Tr}(\gamma^\mu) = 0$

→ We will use three things : (i) $\mathrm{Tr}(A_1A_2\dots A_n) = \mathrm{Tr}(A_2A_3\dots A_nA_1)$

(ii) $\{\gamma^\mu,\gamma 5\} = 0$

(iii) $(\gamma 5)^2 = \mathbb{1}$

$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\mathbb{1}) \overset{(iii)}{=} \mathrm{Tr}(\gamma^\mu\gamma 5\gamma 5) \overset{(ii)}{=} \mathrm{Tr}(\gamma 5\gamma^\mu\gamma 5) = -\mathrm{Tr}(\gamma^\mu\gamma 5\gamma 5)$$

$$= 0.$$

(d) $\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}) = 0 \qquad n \in \text{Odd}.$

$$\rightarrow \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}) = \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\underbrace{\gamma 5\gamma 5}_{\mathbb{1}}) \overset{(iii)}{=} \mathrm{Tr}(\gamma^5\underbrace{\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^5}_{(-1)^n})$$

$$= \mathrm{Tr}(\underset{1}{(-1)^n}\,\gamma 5\gamma 5\,\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n})$$

$$= -1\,\mathrm{Tr}(\dots) = 0\,!$$

(e) $\mathrm{Tr}(\gamma 5) = 0$

$$\rightarrow \mathrm{Tr}(-i\gamma^0\gamma^1\gamma^2\gamma^3) = -i\,\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3) = -i\,\mathrm{Tr}(\gamma^3\underbrace{\gamma^0\gamma^1\gamma^2}_{(-1)^3})$$

$$= -(-i)\,\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3) = -(\mathrm{Tr}(-i\gamma^0\gamma^1\gamma^2\gamma^3))$$

$$= 0$$

□

(f) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

$$\rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\overset{A}{\gamma^\mu}\overset{B}{\gamma^\nu}) + \frac{1}{2}\mathrm{Tr}(\overset{B}{\gamma^\nu}\overset{A}{\gamma^\mu})$$

$$= \frac{1}{2}\mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\}) \overset{(0)}{=} \frac{1}{2}\mathrm{Tr}(2\eta^{\mu\nu}\mathbb{1}_{4\times4}) = \cancel{\frac{2}{2}}\eta^{\mu\nu}\mathrm{Tr}(\mathbb{1}_{4\times4})$$

$$= 4\eta^{\mu\nu}$$

(g) $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu)$

$$\rightarrow \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}((\gamma^\rho)^2\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\rho\overset{(-1)^3}{\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu}) = (-1)^3\,\mathrm{Tr}(\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho)$$

$\rho \neq 0,\mu,\nu$

$$= -\mathrm{Tr}(\underbrace{\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho}_{\text{Cycle}}) = -\mathrm{Tr}(\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu) = 0$$

$\square$

(h) ~~Tr~~ $T^{\mu\nu} = -\frac{i}{2}\gamma^\mu\gamma^\nu$

$$\rightarrow T^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu,\gamma^\nu] = -\frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = -\frac{i}{4}(\underbrace{\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu}_{=1} + \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

$$= -\frac{i}{4}(2\gamma^\mu\gamma^\nu - \{\gamma^\mu,\gamma^\nu\}) = -\frac{i}{2}\gamma^\mu\gamma^\nu + \frac{i}{4}\{\gamma^\mu,\gamma^\nu\}$$

$$= -\frac{i}{2}\gamma^\mu\gamma^\nu + \frac{i}{4}(2\eta^{\mu\nu}\mathbb{1}_{4\times4}) = -\frac{i}{2}\gamma^\mu\gamma^\nu + \frac{i}{2}\eta^{\mu\nu}\mathbb{1}_{4\times4}$$

(i) $\not p\not q = 2p\cdot q\,\mathbb{1}_{4\times4} - \not q\not p$

$$\rightarrow \not p\not q = p_\mu\gamma^\mu q_\nu\gamma^\nu = (\gamma^\mu\gamma^\nu)p_\mu p_\nu = (\gamma^\mu\gamma^\nu + \underbrace{\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu}_{=0})p_\mu p_\nu$$

$$= (\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)p_\mu p_\nu = (2\eta^{\mu\nu}\mathbb{1}_{4\times4} - \gamma^\nu\gamma^\mu)p_\mu p_\nu$$

$$= 2\underbrace{\eta^{\mu\nu}p_\mu p_\nu}_{\text{Dot product}}\mathbb{1} - \underbrace{\gamma^\mu p_\mu}_{\not p}\underbrace{\gamma^\nu p_\mu}_{\not q} = 2p\cdot q\,\mathbb{1}_{4\times4} - \not p\not q$$

$\square$

(j) $\mathrm{Tr}(\not p \not q) = 4p.q = (4\eta^{\mu\nu}p_\mu p_\nu)$

$\rightarrow$ $\mathrm{Tr}(\not p \not q) = \cancel{\mathrm{Tr}(\gamma^\mu p_\mu \gamma^\nu p_\nu)}\ \mathrm{Tr}(\overbrace{2p.q\,\mathbb{1}_{4\times4} - \not q \not p}^{=\not p \not q \text{ from (h)}})$

$= \mathrm{Tr}(2p.q\,\mathbb{1}_{4\times4}) - \mathrm{Tr}(\not q \not p)$

$= 8p.q - \mathrm{Tr}(\not q \not p)$

$= \cancel{8p.q - \mathrm{Tr}(2q.p\,\mathbb{1}_{4\times4} - \not p \not q)}$

~~$\mathrm{Tr}(\not q \not p) =$~~ [illegible]

$\mathrm{Tr}(\not p \not q) + \mathrm{Tr}(\overset{\frown}{\not q \not p}) = 8p.q$

$2\,\mathrm{Tr}(\not p \not q) = 8pq \implies \mathrm{Tr}(\not p \not q) = 4pq.$

(k) $\mathrm{Tr}(\not p_1 \not p_2 \not p_3 \not p_4) = 4[(p_1 p_2)(p_3.p_4) - (p_1.p_3)(p_2.p_4) + (p_2.p_3)(p_1.p_4)]$

$\rightarrow$ $\mathrm{LHS} \equiv \mathrm{Tr}(\gamma^\mu p_{1\mu} \gamma^\nu p_{2\nu} \gamma^\rho p_{3\rho} \gamma^\delta p_{4\delta}) = \mathrm{Tr}(p_{1\mu} p_{2\nu} p_{3\rho} p_{4\delta} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\delta)$

$= p_{1\mu} p_{2\nu} p_{3\rho} p_{4\delta} \mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\delta)$

$\mathrm{Tr}(\underline{\gamma^\mu \gamma^\nu} \gamma^\rho \gamma^\delta) = \mathrm{Tr}[(2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu)\gamma^\rho \gamma^\delta]$

Basic anticommutation relation

$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho \gamma^\delta) - \mathrm{Tr}(\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\delta)$

$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho \gamma^\delta) - \mathrm{Tr}(\gamma^\nu (2\eta^{\mu\rho} - \overset{\rho}{\gamma}\overset{\mu}{\gamma})\gamma^\delta)$ $\quad \mathrm{Tr}(\underline{\gamma^\mu \gamma^\delta} \gamma^\nu \gamma^\rho)$ $\quad \|$

$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho \gamma^\delta) - \cancel{\mathrm{Tr}}\, 2\eta^{\mu\rho}\mathrm{Tr}(\gamma^\nu \gamma^\delta) + \mathrm{Tr}(\gamma^\nu \overset{\rho}{\gamma} \overset{\mu}{\gamma} \gamma^\delta)$

$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho \gamma^\delta) - 2\eta^{\mu\rho}\mathrm{Tr}(\gamma^\nu \gamma^\delta) + \cancel{\mathrm{Tr}((2\eta^{\mu\delta} - \gamma^\delta \gamma^\mu)}$

$+ \mathrm{Tr}((2\underline{\eta}^{\mu\delta} - \underline{\gamma^\delta \gamma^\mu})\gamma^\nu \gamma^\rho)$

$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\rho \gamma^\delta) - 2\eta^{\mu\rho}\mathrm{Tr}(\gamma^\nu \gamma^\delta) + \cancel{\mathrm{Tr}}\, 2\eta^{\mu\delta}\mathrm{Tr}(\gamma^\nu \gamma^\rho)$

$- \mathrm{Tr}(\gamma^\delta \gamma^\mu \gamma^\nu \gamma^\rho)$

$\|$

$\underline{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\delta}$

QFT. Ex. 10
Prob 10.2.
Pg. 3

$$\cancel{2}\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \cancel{2}\eta^{\mu\nu}\,\mathrm{Tr}(\gamma^\rho\gamma^\sigma) - \cancel{2}\eta^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma)$$
$$+\cancel{2}\eta^{\mu\sigma}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho)$$

$$\mathrm{LHS} = p_{1\mu}p_{2\nu}p_{3\rho}p_{4\sigma}\left(\eta^{\mu\nu}\,\mathrm{Tr}(\gamma^\rho\gamma^\sigma) - \eta^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) + \eta^{\mu\sigma}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho)\right)$$
$$\downarrow (*)$$
$$= p_{1\mu}p_{2\nu}p_{3\rho}p_{4\sigma}\left(\eta^{\mu\nu}(4\eta^{\rho\sigma}) - \eta^{\mu\rho}(4\eta^{\nu\sigma}) + \eta^{\mu\sigma}(4\gamma^\nu\gamma^\rho)\right)$$
$$= 4\left[(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$

(l) $\gamma_\mu\not{p}\gamma^\mu = -2\not{p}$

→ $\mathrm{LHS} = \gamma_\mu\gamma^\nu p_\nu\gamma^\mu = p_\nu(\gamma_\mu\gamma^\nu\gamma^\mu)$

$$= p_\nu\left[(\cancel{2} - \gamma^\nu\gamma_\mu + 2\eta_\mu{}^\nu)\right]\gamma^\mu$$
$$= -p_\nu\gamma^\nu\gamma_\mu\gamma^\mu + 2p_\nu\eta_\mu{}^\nu\gamma^\mu$$
$$= -\not{p}(\gamma_\mu\gamma^\mu) + 2p_\nu\gamma^\nu$$
$$= -\not{p}\underbrace{(\gamma_\mu\gamma^\mu)}_{4?} + 2\not{p} = -4\not{p} + 2\not{p} = -2\not{p} \quad \square$$

Proof: $\gamma_\mu\gamma^\mu = 4$!

Start with, $\{\gamma_\mu,\gamma^\mu\} = \cancel{2\eta^\mu{}_\mu}\ \{\gamma_\mu\gamma^\mu + \gamma_\mu\gamma^\mu\} = \underline{2\gamma_\mu\gamma^\mu} = 2\delta^\mu{}_\mu \underline{= 8}$

$$\gamma_\mu\gamma^\mu = 4.$$

(m) $\gamma_\mu\not{p}_1\not{p}_2\gamma^\mu = 4p_1\cdot p_2\,\mathbb{1}_{4\times4}$

~~→ $\mathrm{LHS} = \gamma_\mu\gamma^\nu p_{1\nu}\gamma^\rho p_{2\rho}\gamma^\mu = p_{1\nu}p_{2\rho}\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu$~~

~~$= \cancel{p_{1\nu}p_{2\rho}\gamma_\mu\gamma^\mu\eta^\mu{}_\alpha\gamma} = p_{1\nu}p_{2\rho}\gamma_\mu(\eta^\nu{}_\alpha\gamma^\alpha)(\eta^\rho{}_\beta\gamma^\beta)\gamma^\mu$~~

~~$= p_{1\nu}p_{2\rho}\eta^\nu{}_\alpha\eta^\rho{}_\beta(\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu) \overset{\alpha\to\mu}{\underset{\beta\to\mu}{=}} p_{1\nu}p_{2\rho}\eta^\nu{}_\mu\eta^\rho{}_\mu(\overbrace{\gamma_\mu\gamma^\mu}^{4}\gamma^\mu\gamma^\mu)$~~