Problem 10.1 - Yukawa Potential

Consider non-relativistic fermions coupled to a relativistic real scalar field with the following action

$$S = \int dtd^3x \left\{ -\bar{\psi}\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 - i\epsilon\right)\psi - \frac{1}{2}\phi\left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\phi - g\phi\bar{\psi}\psi \right\}$$

The partition function can be written as

$$Z = \int D\bar{\psi}D\psi D\phi \exp\left[-i\frac{1}{2}\int d^4x\,\phi\left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\phi + i\int d^4xJ\phi + iS_f[\bar{\psi},\psi]\right]$$

where S_f is the free fermionic part of the action and

$$J = -g\bar{\psi}\psi$$

(a) Perform a Gaussian integration of bosonic fields in the partition function and show that the non-relativistic fermions acquire a non-local interaction term in the effective action

$$S = \int dtd^3x \left\{ -\bar{\psi}\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 - i\epsilon\right)\psi \right\} + \frac{g^2}{2}\int d^4xd^4y\bar{\psi}(x)\psi(x)\Delta(x-y)\bar{\psi}(y)\psi(y)$$

where $\Delta(x-y)$ is the real scalar propagator.
Obtain the explicit expression for the propagator $\Delta(x-y)$.
Hint: Write the free part of the scalar field action as $S_\phi = -\frac{1}{2}\int d^4x\, d^4y\, \phi(x)\, \Delta^{-1}(x-y)\, \phi(y)$ and recall the Gaussian integration formulae from one of the previous exercise sheets.

Solution.
We want to show that

$$\int dtd^3x \left\{ -\frac{1}{2}\phi\left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\phi - g\phi\bar{\psi}\psi \right\} = \frac{g^2}{2}\int dtd^3x \left[d^4y\bar{\psi}(x)\psi(x)\Delta(x-y)\bar{\psi}(y)\psi(y)\right]$$

i.e

We know

$$\left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\Delta(x-y) = \delta(x-y)$$

Let us use this definition to get the double integral that we want,

$$\begin{aligned}
\Delta^{-1}(x-y) &= \int dz\, \delta(z-y)\, \Delta^{-1}(x-z) \\
&= \int dz\, \left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\Delta(z-y)\,\Delta^{-1}(x-z) \\
&= \left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right)\delta(x-y)
\end{aligned}$$

Now,

$$\begin{aligned}
&\Rightarrow -\int_x \frac{1}{2}\phi\left(\partial_t^2 - \nabla^2 + M^2 - i\varepsilon\right)\phi - g\phi\bar{\psi}\psi \\
&= -\int_x\int_y \left[\frac{1}{2}\phi(x)\left(\partial_t^2 - \nabla^2 + M^2 - i\varepsilon\right)\delta(x-y)\,\phi(y) - g\phi(x)\,\delta(x-y)\,\bar{\psi}(y)\,\psi(y)\right] \\
&= -\int_x\int_y \left[\frac{1}{2}\phi(x)\,\Delta^{-1}(x-y)\,\phi(y) - g\phi(x)\,\delta(x-y)\,\bar{\psi}(y)\,\psi(y)\right] \qquad (1)
\end{aligned}$$

The first term is in the form we would like, now let us manipulate the second term. For this we will use

$$\delta(x-y) = \int_z \Delta^{-1}(x-z)\,\Delta(z-y)$$

in

$$\int_x \int_y g\phi(x)\,\delta(x-y)\,\bar{\psi}(y)\,\psi(y)$$

giving us

$$\Rightarrow \int_x \int_y \int_z g\phi(x)\,\Delta^{-1}(x-z)\,\Delta(z-y)\,\bar{\psi}(y)\,\psi(y)$$

Plug this in (1),

$$\Rightarrow -\int_x \int_y \left[\frac{1}{2}\phi(x)\,\Delta^{-1}(x-y)\,\phi(y) - g\phi(x)\,\delta(x-y)\,\bar{\psi}(y)\,\psi(y)\right]$$
$$= -\int_x \int_y \left[\frac{1}{2}\phi(x)\,\Delta^{-1}(x-y)\,\phi(y) - \int_z g\phi(x)\,\Delta^{-1}(x-z)\,\Delta(z-y)\,\bar{\psi}(y)\,\psi(y)\right]$$

Now switch some of the integration variables to our convenience,

$$\Rightarrow -\int_x \int_z \left(\frac{1}{2}\phi(x)\,\Delta^{-1}(x-z)\,\phi(z)\right) - \int_x \int_y \int_z g\phi(x)\,\Delta^{-1}(x-z)\,\Delta(z-y)\,\bar{\psi}(y)\,\psi(y)$$
$$= -\int_x \int_z \phi(x)\left[\left(\frac{1}{2}\Delta^{-1}(x-z)\,\phi(z)\right) - \int_x \int_y \int_z g\,\Delta^{-1}(x-z)\,\Delta(z-y)\,\bar{\psi}(y)\,\psi(y)\right]$$

Completing square is what has to be done after this, this is a little unclear to me as of now, I am going to assume the result is correct and move on to other parts of the problem

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Assuming we have proved what we wanted, we now have our total action as

$$S = \int dt d^3x \left\{-\bar{\psi}\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 - i\epsilon\right)\psi\right\} + \frac{g^2}{2}\int d^4x d^4y \bar{\psi}(x)\psi(x)\Delta(x-y)\bar{\psi}(y)\psi(y)$$

(Here, the interaction term is non-local because, it does not have to be at a singular spacetime point - in that case it would have a delta-function. Instead, it has a propagator which strengthens our argument of non-locality even more)

Now, we still have to find $\Delta(x-y)$, assuming we know $\Delta^{-1}(x-y)$(Which we do, its the Klein Gordon operator for bosonic fields and the Dirac operator for fermionic fields, in our case it is the Klein-Gordon operator, as we did the previous simplification for the Bosonic fields with the interaction term)

$$\delta^4(x-z) = \int_y \Delta^{-1}(x-y)\,\Delta(y-z)$$
$$= \int_y \underbrace{\left(\partial_t^2 - \nabla^2 + M^2 - i\varepsilon\right)\delta^4(x-y)}_{=\Delta^{-1}} \underbrace{\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(y-z)}\Delta(p)}_{\Delta \text{in momentum space}}$$
$$= \left(\partial_t^2 - \nabla^2 + M^2 - i\varepsilon\right)\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-z)}\Delta(p)$$
$$= \int \frac{d^4p}{(2\pi)^4}\,\left(\partial_t^2 - \nabla^2 + M^2 - i\varepsilon\right) e^{ip\cdot(x-z)}\Delta(p)$$
$$= \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-z)}\,\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right)\Delta(p)$$

Now, expand the LHS in Fourier space,

$$\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-z)} = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-z)}\,\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right)\Delta(p)$$

giving us

$$\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right) \Delta(p) = 1$$
$$\Delta(p) = \frac{1}{\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right)}$$

Finally, we can write this in position space as

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\, \Delta(p)$$
$$= \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\, \frac{1}{\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right)}$$

(b) The new interaction term for fermions is non-local in time due to the fact that the boson was a relativistic one. The potential in this case is a retarded one (like the Lienard-Wiechert potential from classical electromagnetism). However, since we are considering non-relativistic fermions we are interested in the non-relativistic limit of the potential, i.e. instantaneous interactions.
The limit can be performed by restoring the speed of light in our Laplacian

$$\partial_t^2 - \nabla^2 \to \frac{1}{c}\partial_t^2 - \nabla^2$$

and taking $c \to \infty$, i.e we can neglect the terms associated with the time-derivative.
Show that in this case the propagator,

$$\Delta(x-y) = \delta\left(x^0 - y^0\right) \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{\vec{p}^2 + M^2}$$

becomes local in time and consequently the interaction is local in time.

Solution.

$$\Delta(x-y) = \underbrace{\int \frac{dp^0}{(2\pi)}\, e^{ip\cdot\left(x^0-y^0\right)}}_{=\delta(x^0-y^0)} \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\left(-p_0^2 + \vec{p}^2 + M^2 - i\varepsilon\right)}$$

Getting rid of the ∂_t in the inverse propagator is equivalent to getting rid of the p_0 term in the propagator, giving us

$$\Delta(x-y) = \delta\left(x^0 - y^0\right) \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\left(\vec{p}^2 + M^2 - i\varepsilon\right)}$$

which is what we wanted to show.

(c) In the instantaneous approximation the effective fermion vertex can be written as

$$\int dt \int d^3x\, d^3y\, \bar{\psi}(t,\vec{x})\, \psi(t,\vec{x})\, V_2(\vec{x}-\vec{y})\, \bar{\psi}(t,\vec{y})\, \psi(t,\vec{y})$$

where we have defined the two body potential as

$$V_2(\vec{r}) = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p})^2 + M^2} e^{i\vec{p}\cdot\vec{r}}$$

Compute $V_2(\vec{r})$.

Solution.

Let us go to spherical coordinates $(|\vec{p}| = p, \theta, \varphi)$, $r = |\vec{r}|$

$$V_2(\vec{r}) = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p})^2 + M^2} e^{i\vec{p}\cdot\vec{r}}$$
$$= \frac{g^2}{2(2\pi)^3} \int_0^\infty dp \int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \frac{p^2}{p^2 + M^2} e^{ip\,r\,\cos\theta}$$

The extra p^2 came from the Jacobian,

$$= \frac{g^2}{2(2\pi)^2} \int_0^\infty dp \frac{p^2}{p^2+M^2} \int_{-1}^{1} d\cos\theta\, e^{ip\,r\,\cos\theta}$$
$$= \frac{g^2}{2(2\pi)^2} \int_0^\infty dp \frac{p^2}{p^2+M^2} \left[\frac{e^{ip\,r\,\cos\theta}}{ipr}\right]_{\cos\theta=-1}^{\cos\theta=1}$$
$$= \frac{g^2}{2(2\pi)^2} \int_0^\infty dp \frac{p}{p^2+M^2} \left[\frac{e^{ip\,r} - e^{-ip\,r}}{ir}\right]$$
$$\clubsuit = \frac{g^2}{8\pi^2 ir} \left[\int_{-\infty}^{0} dp \frac{p}{p^2+M^2} e^{-ipr} + \int_0^\infty dp \frac{p^2}{p^2+M^2} e^{ipr}\right]$$
$$(p \to -p) = \frac{g^2}{8\pi^2 ri} \int_{-\infty}^{\infty} dp \frac{p}{p^2+M^2}\, e^{-ipr}$$
$$\spadesuit = \frac{g^2}{8\pi^2 ri} \left(-i\pi e^{-Mr}\right)$$
$$= \frac{-g^2}{8\pi} \frac{e^{-M|\vec{r}|}}{|\vec{r}|} \qquad \text{(Yukawa potential)}$$

The ♣ in the above calculation is a little unclear to me. As far as I understand from the solution set, $p > 0$ needs to be satisfied, so in order to maintain that we do this. The ♠ arrives from Cauchy integral. In principle when I attempted the calculation I stopped on the line before ♣ and just used Mathematica.

(d) What is the V_2 potential if the boson is massless $M = 0$?

Solution.
For the massless case, take the $M \to 0$ in the Yukawa potential calculated above, giving us,

$$-\frac{g^2}{8\pi} \frac{1}{|\vec{r}|}$$

which is the form of a Coulomb potential.