Ordinary Differential Equations

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Definition 1. *Differential equation*

A differential equation is basically a mathematical equation that relates some function with its derivatives.

1 Classification

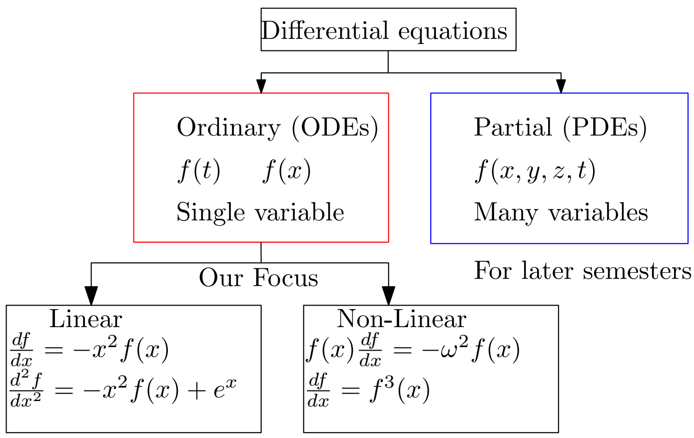

Figure 1.1.

Example 1.1.

1. Newton's second law for a ball falling under gravity(For simplicity purposes let's consider 1D):

$$\begin{aligned} F &= ma \\ -mg &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \end{aligned}$$

   Linear ODE of second order in t

2. Newton's second law for a mass on a spring :

$$\begin{aligned} F &= ma \\ -kx &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \end{aligned}$$

   Linear ODE of second order in t.

3. Gauss law (First Maxwell's equation in differential form):

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

   PDE of first order in x, y and z.

4. Schrodinger's equation for a free particle (Time dependent) :

$$\begin{aligned} i\hbar\frac{\partial\Psi(x,t)}{\partial t} &= \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}+V\right)\Psi(x,t) \\ &= \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\right)\Psi(x,t) \end{aligned}$$

   PDE of second order in x and first order in t.

5. Schrodinger's equation for a free particle (Time independent) :

$$\begin{aligned} \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}+V\right)\psi(x) &= E\psi(x) \\ \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\right)\psi(x) &= E\psi(x) \end{aligned}$$

   ODE of second order in x.

In the above example we have listed few instances of the most fundamental equations of classical mechanics, quantum mechanics, electrodynamics which are basically the most fundamental building blocks of Physics in all of its generality. All of the equations above are differential equations. Not necessarily of the same type though. Depending on the type and order of the differential equation, there are different techniques to solve them.

Definition 1.1. *Linear ODE*

Each term contains none or exactly one of the dependent variable or its derivatives.

- *No mixed product like* $f(x)\frac{\mathrm{d}f}{\mathrm{d}x}$ *, etc.*
- *Each term containing* f *has at most one power. That means* $f^n(x)$ *for anything* $n > 1$ *is forbidden.*

Definition 1.2. *Order of an ODE.*

The largest number of derivatives taken is defined as the order of the ODE.

Example.

$$f(x)\frac{\partial f}{\partial x} = 0 \quad \rightarrow \quad \text{First order non linear ODE}$$
$$\frac{\partial^2 f}{\partial x^2} = \sin x \quad \rightarrow \quad \text{Second order non linear ODE.}$$

Example 1.2. Blood alcohol.

Suppose you are at a bar and drinking at a constant rate of d (Drunk's constant). Your body detoxifies alcohol linearly proportionate to the amount of alcohol in your blood.

Write a differential equation out of this statement for the change of alcohol in your blood over time.

Solution.

Let f be the amount of alcohol in your blood. We want to know how does f change over time.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = d - \alpha y$$

Analysing the equation :

- RHS : The rate at which the amount of alcohol changes over time.
- LHS : It changes as you are drinking and it reduces depending on y (How much alcohol you have in your blood) with a proportionality factor α.

2 First order ODEs (Linear)

First order linear ODEs are the most simplest type of ODEs. They are also divided into two groups based on $q(x)$ in this equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y^1 + q(x)y^0$$

1. If $q(x) = 0 \longrightarrow$ `Homogeneous`

2. If $q(x) \neq 0 \longrightarrow$ `Inhomogeneous`

2.1 Homogeneous first order linear ODE.

General form :

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y$$

Technique to solve them : Seperation of variables

1. Bring x's to one side and all y's to the other.

$$\frac{\mathrm{d}y}{y} = p(x)\,\mathrm{d}x$$

2. Integrate :

$$\begin{aligned}
\int \frac{1}{y}\mathrm{d}y &= \int p(x)\mathrm{d}x \\
\ln(y) &= \int p(x)\mathrm{d}x + c \\
y &= e^{\int p(x)\mathrm{d}x + c} = e^{\int p(x)\mathrm{d}x} e^{c} = c_1 e^{\int p(x)\mathrm{d}x}
\end{aligned}$$

Example 2.1. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2y$. For $y(0) = 1$(This is called an initial condition)

Solution.

$$\begin{aligned}
\frac{\mathrm{d}y}{y} &= 3x^2\mathrm{d}x \\
\ln(y) &= x^3 + c \\
y &= Ae^{x^3}
\end{aligned}$$

How to find A? Well, use the remaining data given in the problem.

$$\begin{aligned}
y(0) &= 1 \\
A &= 1
\end{aligned}$$

There we have

$$y = e^{x^3}$$

as a solution for this ODE with this initial condition.

Example 2.2. (Homework)

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (xe^x)y$ for $y(0) = 1$.

Solution. $y = e^{e^x(x-1)}$