A Lagrangian problem

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Problem 1. [2]

Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other as shown in the figure below. The bottom end of the lower stick is hinged at the ground. They are held such that lower stick is vertical, and the upper one is tilted at a small angle φ with respect to the vertical. At the instant they are released, what are the angular accelerations of the two sticks? (Use small angle approximation for φ)

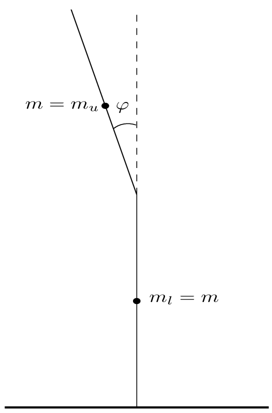

Figure 1.

Solution. Let us setup our diagram as follows (Just after we left the top stick) :

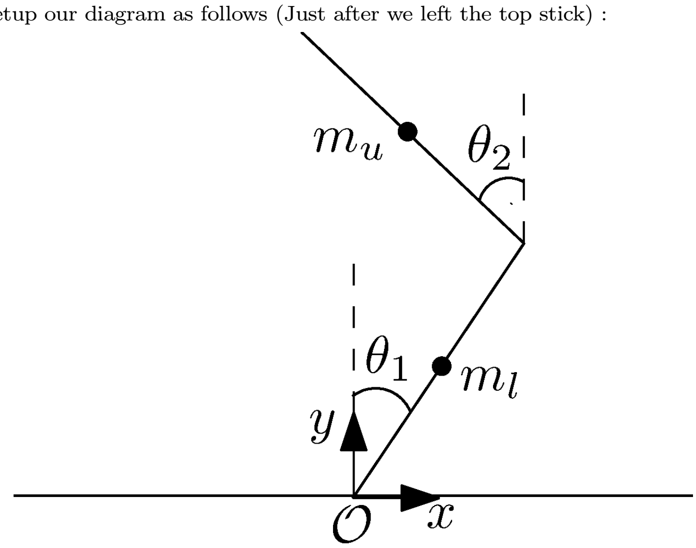

Figure 2.

Let us call the lower mass as m_l and the upper mass as m_u just for our convinience (Remember they have the same mass). Let $\vec{x}_l$ and $\vec{x}_u$ be the position vectors of the lower and the upper mass respectively. Take the origin of your coordinate system to be as defined.

Let us look at the two position vectors of the two masses :

$$\vec{x}_l = \begin{pmatrix} r\sin\theta_1 \\ r\cos\theta_1 \end{pmatrix}$$

$$\vec{x}_u = \begin{pmatrix} 2r\sin\theta_1 - r\sin\theta_2 \\ 2r\cos\theta_1 + r\cos\theta_2 \end{pmatrix}$$

Now for the lagrangian we need the potential energy and the kinetic energy (V and T respectively for the two masses). For the kinetic energy we need the derivative of the position squared.

$$\begin{aligned} \dot{\vec{x}}_l &= \begin{pmatrix} r\cos\theta_1\,\dot{\theta}_1 \\ -r\sin\theta_1\,\dot{\theta}_1 \end{pmatrix} \\ \dot{\vec{x}}_u &= \begin{pmatrix} 2r\cos\theta_1\,\dot{\theta}_1 - r\cos\theta_2\,\dot{\theta}_2 \\ -2r\sin\theta_1\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2 \end{pmatrix} \end{aligned}$$

Now the kinetic energies of the two particles :

$$\begin{aligned} T_l &= \frac{1}{2}m_l\,(\dot{\vec{x}_l})^2 \\ &= \frac{1}{2}(r^2\cos^2\theta_1\,(\dot{\theta}_1)^2 + r^2\sin^2\theta_1\,(\dot{\theta}_1)^2) \\ &= \frac{1}{2}mr^2\,(\dot{\theta}_1)^2 \\ T_u &= \frac{1}{2}m_u(\dot{\vec{x}}_u)^2 \\ &= \frac{1}{2}mr^2\,((2\cos\theta_1\dot{\theta}_1 - \cos\theta_2\,\dot{\theta}_2)^2 + (-2r\sin\theta_1\dot{\theta}_1 - r\sin\theta_2\,\dot{\theta}_2)^2) \end{aligned}$$

T_l is quite easy to deal with. But knowing that E-L equations are a bunch of differential equations having non-linear terms is going to make life hard. So we use the fact that we started with a small angle ε and immidiately after releasing the upper stick, the angle still remain small. We need to calculate the angular acceleration of these sticks just after they are released so this is quite a reasonable assumption. So in the small angle approximation T_l stays the same but the $\sin\theta$ terms become θ and the $\cos\theta$ terms become $1-\frac{\theta^2}{2}$ as we only drop terms higher than second order. But if we see carefully for T_u the terms in theta for sin and cos will be squared. So they are the sin and cos squared of a small angle. So we can basically neglect sin and take cos as 1.

So the new T_u which we are going to use is :

$$\begin{aligned} T_u &= \frac{1}{2}mr^2\,((2\dot{\theta}_1 - \dot{\theta}_2)^2 \\ T_l &= \frac{1}{2}mr^2\,(\dot{\theta}_1)^2 \\ T = T_u + T_l &= \frac{1}{2}mr^2(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \end{aligned}$$

Now we use the same approximations in for the Potential energy, the first one, not the one where we drop sin and take cos as one.

$$\begin{aligned} V = V_l + V_u &= mgr\cos\theta_1 + (2mgr\cos\theta_1 + mgr\cos\theta_2) \\ &= mgr\,(3\cos\theta_1 + \cos\theta_2) \\ &\quad \text{(After using approximation)} \\ &= mgr\left(3\left(1-\frac{\theta_1^2}{2}\right) + 1 - \frac{\theta_2^2}{2}\right) \\ &= mgr\left(4 - \frac{3\theta_1^2}{2} - \frac{\theta_2^2}{2}\right) \end{aligned}$$

So now we have the Lagrangian as :

$$\begin{aligned} L &= T_l + T_u - V \\ &\approx \frac{1}{2}mr^2(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) - mgr\left(4 - \frac{3\theta_1^2}{2} - \frac{\theta_2^2}{2}\right) \end{aligned}$$

Now plug the lagrangian for the two E-L equations :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = \frac{\partial L}{\partial\theta_1} \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) = \frac{\partial L}{\partial\theta_2} \tag{2}$$

Do the computation by plugging in the L and you get the following equations :

$$\begin{aligned} 5\ddot{\theta}_1 - 2\ddot{\theta}_2 &= \frac{3\,g}{r}\theta_1 \\ -2\,\ddot{\theta}_1 + \ddot{\theta}_2 &= \frac{g}{r}\,\theta_2 \end{aligned}$$

Now we need some initial conditions to make some sense out of these equations. We know that the instant the sticks were released $\theta_1 = 0$ and $\theta_2 = \varphi$. Plugging this in the above equations we get :

$$\begin{aligned} 5\ddot{\theta}_1 - 2\ddot{\theta}_2 &= 0 \\ -2\,\ddot{\theta}_1 + \ddot{\theta}_2 &= \frac{g}{r}\varphi \end{aligned}$$

Solving these we get :

$$\begin{aligned} \ddot{\theta}_1 &= \frac{2g}{r}\varphi \\ \ddot{\theta}_2 &= \frac{5\,g}{r}\,\varphi \end{aligned}$$

Bibliography

[1] https://www.informatik.hu-berlin.de/de/forschung/gebiete/ki/lehre/ws0506/kogrob/bioloid_sim/material/lagrangian_mechanics.pdf, ().

[2] David Morin. *Introduction to Classical Mechanics: With Problems and Solutions*. Cambridge.