MOCK EXAM I.1

Freshmen Tutorial Winter Semester 2018/19
Theoretical Physics I.1, Mathematics I.1, Experimental Physics I.1

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Full Name	Matriculation no.

Subject	TPI.1	MI.1	EPI.1	**Total** ↓
Marks				
Out of	30	30	30	90
Grade				

Information Regarding the Exam

- The exam has three sections. The exam will be held in following order
  - Theoretical Physics I (5:15 PM to 6:00 PM) (45 Minutes/30 Points)
  - Mathematics I (6:05 PM to 6:50 PM) (45 Minutes/30 Points)
  - Experimental Physics I (6:55 PM to 7:40 PM) (45 Minutes/30 Points)
- For Theoretical Physics I and Mathematics I you will be allowed to use the following objects (Calculator permitted only for Experimental Physics I) :
  1. A physical writing instrument. (Pen, Pencil, etc)
  2. Empty sheets of paper. (Blank/Ruled/Chequered/Dotted)
  3. Your own brain.
- Use of all electronic devices is strictly prohibited. Anyone using any digital devices will be barred from the exam. (No smart watches.)
- No communication during the exam. If you have any questions just raise your hand.

. ***You need 50% in all the three sections to pass the exam.***[1]

1. Unless for any of the three sujects the average score falls below 50%. Then the passing percentage will be recalculated accordingly. (So basically, don't give up! You could score really decent assuming the fact that even others find the exam difficult.)

Section I : Theoretical Physics I.1

Full Name	Matriculation Number

QS	1.1	1.2	1.3*	1.4	1.5	Total
Points gained						
Max points	4	8	7	6	5	**30 points**

Problem 1.1. Vector Algebra

For the given figure below which is a cube, **{4p}**

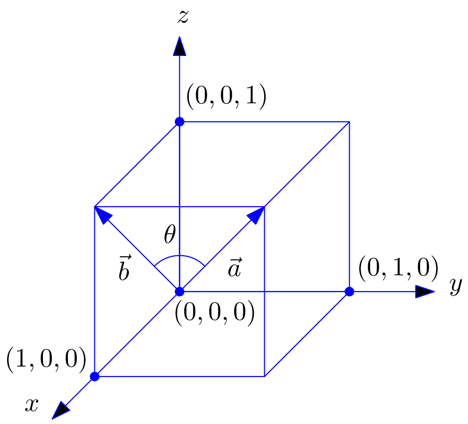

Figure 1.1.

i. Write down the expression of the position vectors $\vec{a}$ and $\vec{b}$ with respect to the origin. **[0.5p]**

ii. Calculate the angle θ which is formed by the diagonals $\vec{a}$ and $\vec{b}$. **[2p]**

iii. Calculate the volume of the pyramid formed by the vectors $\vec{a}, \vec{b}$ and the vector from the origin to $(0,0,1)$.

[1.5p]

Problem 1.2. Understanding Jacobians.*

For the given coordinate system transformation answer the following questions : **{8p}**

$$
\begin{aligned}
(x,y,z) &\mapsto (\rho,\varphi,z)\\
x &= \rho\cos\varphi\\
y &= \rho\sin\varphi\\
z &= z\\
\text{Where:}&\\
\rho &= \sqrt{x^2+y^2}\\
\varphi &= \text{Polar angle (Angle in the } xy \text{ plane, from the } x \text{ axis)}
\end{aligned}
\tag{1.1}
$$

1. Sketch a qualitative diagram for this coordinate system with the following features. **[3p]**
   i. Label the coordinate system with it's actual name. [0.5p]
   ii. A position vector to a random point $\vec{p} = (p_x, p_y, p_z)$ [0.5p]
   iii. Label all the 5 quantities which are given in the problem. (Focus on labeling the right hand side of the first three equations.) [2p]

   (Hint: Draw a right handed cartesian xyz coordinate system and a position vector to some random point in the first octant.)

2. Jacobian : **[3p]**
   i. Calculate the Jacobian. [2P]
   ii. Using the Jacobian write down how the *volume element* looks like in this coordinate system. [1P]
3. Now ignore the z coordinate, **[2p]**
   i. What is the name of the coordinate system is (ρ, φ)? [0.5P]
   ii. Sketch how the grid looks like in this coordinate system (ρ, φ). [1.5P]

   (Basically the sketch the grid of the xy plane for (ρ, φ) as we are ignoring the z coordinate.)

Problem 1.3. $\vec{\nabla}$ operator in arbitrary coordinate system.[1.1]
Use equation(1.1) as your coordinate system, but you do not need any results from the previous problem in order to be able to solve this. **{7p}**

The $\vec{\nabla}$ operator stays the same expression when you take the gradient, divergence or curl in cartesian coordinate system. In other coordinate systems it changes the expression for gradient,divergence and curl according to their Jacobians with respect to cartesian coordinate system.

From the information above we know $\vec{\nabla}^{(\text{cart})}_{\text{grad}} = \vec{\nabla}^{(\text{cart})}_{\text{div}} = \vec{\nabla}^{(\text{cart})}_{\text{curl}}$ in cartesian coordinate system. But in any other coordinate system, suppose ($w\ t\ f$ coordinate system) $(x, y, z) \mapsto (\omega, \tau, \varphi)$ it changes using this formula :

$$\vec{\nabla}^{(wtf)}_{\text{grad}} = \left(\frac{1}{h_1}\frac{\partial}{\partial\omega}, \frac{1}{h_2}\frac{\partial}{\partial\tau}, \frac{1}{h_3}\frac{\partial}{\partial\varphi}\right) \tag{1.2}$$

where h_1, h_2, h_3 are called scale factors and are defined as follows :

$$h_1 = \left|\frac{\partial\vec{r}}{\partial\omega}\right| \quad h_2 = \left|\frac{\partial\vec{r}}{\partial\tau}\right| \quad h_3 = \left|\frac{\partial\vec{r}}{\partial\varphi}\right|$$

where $\vec{r} = \vec{r}(\omega, \tau, \varphi) = (r_1(\omega, \tau, \varphi), r_2(\omega, \tau, \varphi), r_3(\omega, \tau, \varphi))$ is the position vector.

Using all the explaination above answer the following questions :

i. How does a position vector look like in the coordinates system 1.1? [1p]

ii. Calculate the $\vec{h}_1, \vec{h}_2, \vec{h}_3$. [3p]

iii. How does the gradient operator look like in 1.1 ? [3p]
(Hint : Calculate h_i's for the position vector in 1.1 and then just plug them in 1.2)

Problem 1.4. Using the following given field answer the questions **{6p}**
(Choose the correct coordinate system to do your computations.):

$$\begin{aligned}\varphi(x, y, z) &= 2x\, z\, e^{xz} \\ \vec{A} &= (xy, yz, zx)\end{aligned}$$

1. Choose the appropriate field from the given fields to take the gradient and evaluate it at $(1, 1, 1)$. **[2p]**
2. Take the divergence of the gradient of the field at $(-1, 1, 0)$ **[2p]**
3. Choose the appropriate field from the given fields to take the curl and evaluate it at (1,0,1)**[2p]**

Problem 1.5. Prove the following identities using the index notation and levi civita tensor. **{5P}**

1. $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$ [5p]

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1.1. Read it carefully, this problem tests how well you can read, understand and implement mathematical texts. Everything that you need is given in the problem.

Section II: Mathematics I.1

Full Name	Matriculation Number

QS	2.1	2.2*	2.3	2.4	2.5	2.6	2.7	Total
Points gained								
Max points	6	8	4	3	2+*4*	4	3	**30 points**

Problem 2.1. Set theory and real sequences. (Short questions) **{6p}**

1. Define what does it mean for a set to be countably infinite. **[1p]**
2. Is the set of rational numbers $\mathbb{Q}$ countable? Justify your claim in a few words. **[1p]**
3. Complete the sentence while defining the condition mathematically. **[4p]**
   i. A real sequence (x_n) is said to be **convergent** if ___________. [2p]
   ii. A real sequence (x_n) is called a **cauchy sequence** if _________. [2p]

Problem 2.2. Topology.* [2.1] **{8p}**

If I have a set M and $A \subseteq \mathcal{P}^M$ (where $\mathcal{P}^M$ is known as the power set of M, it is a set containing all the subsets of M). So A (which is a subset of the power set of M) is called a **Topology** of M (denoted by (M, A)) if A satisfies these three conditions :

i. $M, \varnothing \in A$
   (Read as : M and the null set ($\varnothing$) belong to A)
ii. $\forall U \in A$ and $\forall\ V \in A$ then $\Longrightarrow U \cap V \in A$
iii. $\forall U \in A$ and $\forall V \in A$ then $\Longrightarrow U \cup V \in A$ (This works for countably infinite unions.)

Based on the following information given to you, answer the following questions :

1. If $W \in A$ (That is W is an element of the set A) what mathematical object is it? **[1p]**
2. Suppose I have a set $M = \{1, 2, 3\}$: **[4p]**
   a. Is $\Theta_1 = \{\varnothing, \{1, 2, 3\}\}$ a topology on M ? Justify. [2p]
   b. Is $\Theta_2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ a topology on M? Justify. [2p]
3. For an arbitrary set M, you can always define two topologies. **[1p]**
   a. Topology with fewest elements possible ($\Theta_{\min}$). Define it. [0.5p]
   b. Topology with maximum elements possible ($\Theta_{\max}$). Define it. [0.5p]
4. Define a topology on the set $M = \{1, 2, 3\}$ which is **not** $\Theta_{\min}$ or $\Theta_{\max}$. **[2p]**

Problem 2.3. Functions. **{4p}**

Find the domain and range of the following functions.

1. $f(x) = \dfrac{1}{\sqrt{3 + x}}$ **[2p]**
2. $f(x) = \sin x$ **[2p]**

2.1. (This problem is quite abstract but everything that you need to solve it except really elementary set theory is in the question.)

Problem 2.4. Definition of limit.
Using the $\varepsilon - \delta$ definition of the limit, prove that $\lim_{x\to -1}(3x+5) = 2$ **{3p}**

Problem 2.5. **{2p}**
Compute the limits if they exist or prove that they do not exist. **+*4*p Bonus**
(**Without using L'Hopital 's rule)**

1. $\lim_{x\to 2} \dfrac{x^2-4}{x^2+2x-8}$ **[2p]**
2. $\lim_{x\to +\infty} \sqrt{x+3} - \sqrt{x}$. Hint : Use the expansion of (a^2-b^2). [Bonus *2*p]
3. $\lim_{x\to 0} \cos\left(\dfrac{1}{x^2}\right)$. [Bonus *2*p]

Problem 2.6. Definitions. **{4p}**
Define mathematically: (If you define in words you may get quarter points from the total and some mathematical buildup may entitle you to a few more fractional points.)
(**Define does not mean give examples. You have to define the given mathematical objects in an abstract setting. But if you want you can give examples just to show you know it well.)**

a) Supremum of a set. **[2p]**

b) Bijective map. **[2p]**

Problem 2.7. Induction **{3p}**
For which $n \in \mathbb{N}$ does this inequality hold?

$$2^n > 2n+1 \tag{2.1}$$

Prove it.

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Section III: Experimental Physics I.1

Full Name	Matriculation Number

QS	3.1	3.2*	3.2	3.3	Total
Points gained					
Max points	7	8+*3*	8	7	**30 points**

Problem 3.1. 1D Motion **{7p}**

Given a position function for a particle moving in one dimension :

$$\begin{aligned} x(t) &= At^2+Bt+C \\ A,B,C &\in \mathbb{R}\,(\text{Constants}) \\ \text{Initial conditions} &\rightarrow x(0)=4\,[m] \\ & \quad\; v(0)=5\,[m/s] \\ & \quad\; a(0)=g\,[m/s^2] \end{aligned}$$

1. Find $v(t)$ and $a(t)$. (Velocity and acceleration.) **[1p]**
2. Find the constants A,B and C. **[1.5p]**
3. Write the dimensions of A,B,C. **[1.5p]**
4. Calculate the position, velocity and acceleration of the particle as $t=2$. **[1.5p]**
5. Plot graphs for $x(t), y(t), z(t)$. **[1.5p]**

Problem 3.2. Again 1D motion.*

A beach-ball is dropped from rest at height h. Assume that the drag force from the air takes the form $F_d=-m\alpha v$. Follow the procedure step by step to calculate the $y(t)$ for the ball. (Just consider motion in one direction.)

1. Why am I using $y(t)$ and not $x(t)$ like in the previous problem? Is it okay if I use $x(t)$ instead? **[1p]**
2. Define your coordinate system and write down the equation of motion in the following form :

$$F=F_d+\sum_i F_{\text{of}_i} \tag{3.1}$$

   where $\sum F_{\text{of}_i}$ stands for the forces other than F_d acting on the ball. (There could be more than one force so the subscript i stands for $F_{\text{of}_1}, F_{\text{of}_2}, \ldots$ which where F_{of_1} is read as other force 1, F_{of_2} is read as other force 2 and so on and so forth.) **[1p]**
3. After substituting the appropriate forces with correct signs, replace the left hand side of the equation(3.1) with $F=ma=m\dfrac{\mathrm{d}v}{\mathrm{d}t}$. Write down your equation and rearrange it in the following form,

$$m\mathrm{d}v = \left(F_d+\sum_i F_{\text{of}_i}\right)\mathrm{d}t \tag{3.2}$$

   [1p]
4. Now bring equation 3.2 into the following form :

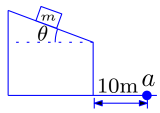
$$\frac{m\mathrm{d}v}{f(v)} = f(t)\mathrm{d}t \tag{3.3}$$

   This step is called seperation of variables. Basically bring all the variables containing v and t together on the right hand side of equation(3.2) and name them $f(v)$ and $f(t)$ respectively. Now rearrange them as shown in equation(3.3). **[2p]**
5. Integrate this to get $v(t)$ with the appropriate limits of integration and solve for $v(t)$ **[3p]**

6. Integrate $v(t)$ using the same procedure as in point 4 (Now it will be for $f(y)$ instead of $f(v)$) and solve for $y(t)$ [***Bonus 3p***]

Problem 3.3. Block falling down an incline. **{8p}**

A block of mass m starts from rest and slides frictionless down an incline at an angle of $\theta = 30°$ as show in the figure. Calculate the following things :



1. Before starting calculations, draw a free body diagram for the block. **[2p]**
2. Acceleration of the block while it is in contact with the incline. **[2p]**
3. The block leaves the incline with a speed of $10\frac{\mathrm{m}}{\mathrm{s}}$. Calculate the length of the incline. **[2p]**
4. The block lands at a, how long does it take the block to fall from the edge to the floor? **[2p]**

Problem 3.4. Bottle throw. **{7p}**

A bottle is thrown horizontally from a train at right angle to the train's velocity. It hits the ground at a point that is 5m below, 10m away (in perpendicular direction) and 25m away (in the direction along the velocity of the train) from the point of throw. Calculate the following things:

1. Before calculating anything, Choose an appropriate coordinate system and draw a quick sketch of it. **[1p]**
2. The speed of the train. **[2p]**
3. The initial velocity and speed of the bottle. **[2p]**
4. The velocity and speed of the bottle as it hits the ground. **[2p]**

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